

**Giornata degli attuari delle pensioni**  
**Gruppo di lavoro dei percettori pensioni/rendite**

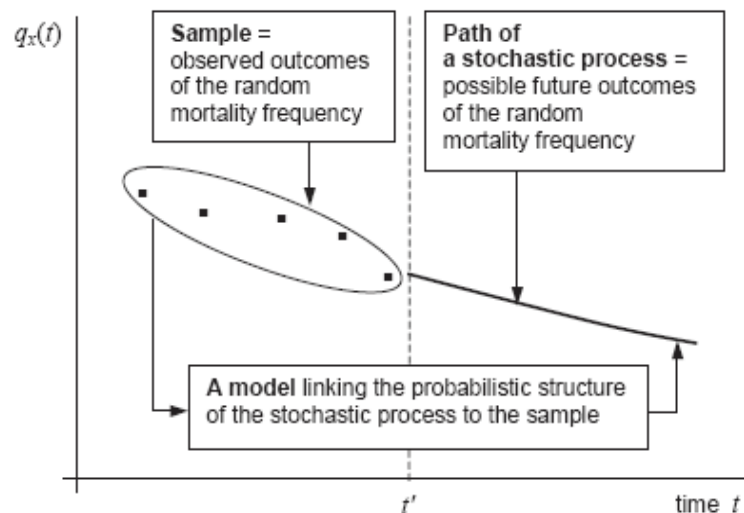
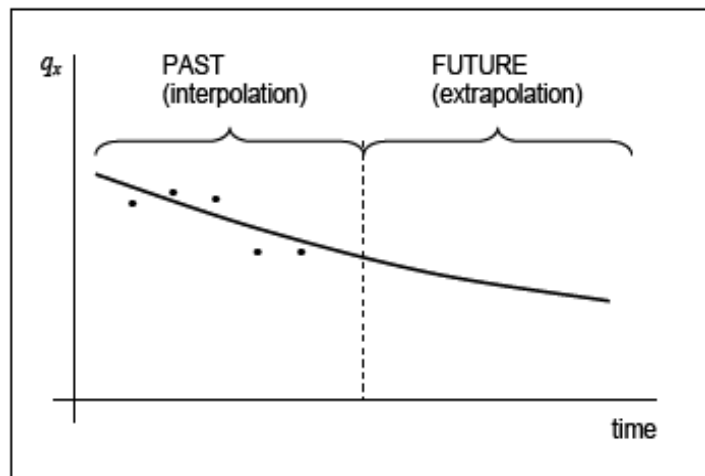
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**I modelli di proiezione della mortalità**

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# I modelli di proiezione della mortalità



Fonte: Olivieri - Pitacco "Life annuities and longevity dynamics" WP n. 36, CERAP 2006

## Metodi estrapolativi

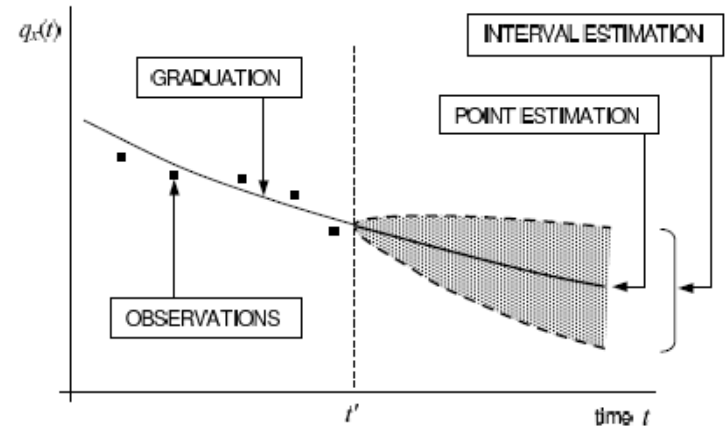
- Interpolazione dei trend di mortalità osservati in passato
- Ipotesi: I trend osservati si ripeteranno in futuro → estrapolazione dei trend
- La natura stocastica della mortalità non viene considerata

## Modelli stocastici

- I tassi di mortalità osservati sono estrazioni di variabili casuali che rappresentano la mortalità passata
- I tassi di mortalità proiettati sono stime di variabili casuali che rappresentano la mortalità futura
- Si definiscono un insieme di ipotesi circa la mortalità e un legame tra osservazioni e proiezioni

# I modelli stocastici di proiezione della mortalità

- In un modello stocastico i risultati delle proiezioni forniscono sia stime puntuali che intervalli di confidenza dei futuri tassi di mortalità



Fonte: Olivieri - Pitacco "Life annuities and longevity dynamics" WP n. 36, CERAP 2006

- Le principali caratteristiche di uno scenario di mortalità possono essere ben rappresentate
- Esempi di leggi di mortalità: Gompertz, Makeham, Weibull, Heligman-Pollard

- Modello Lee-Carter e sue estensioni → E' un esempio significativo di approccio stocastico nella proiezione della mortalità
- Il tasso centrale di mortalità è modellizzato come un processo stocastico

Modelli basati su leggi di mortalità

Modelli distribution-free

# Il modello Lee-Carter

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- ▶ I tassi centrali di mortalità sono rappresentati da:

$$\log(m_{x,t}) = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)} + \varepsilon_{x,t}$$

- ▶ con errori  $\varepsilon_{x,t}$  i.i.d. con distribuzione  $N(0, \sigma_\varepsilon)$

- ▶ Parametri individuati sub vincoli:  $\sum_t k_t^{(2)} = 0$  ;  $\sum_x \beta_x^{(2)} = 1$

- ▶  $k_t^{(2)}$  processo stocastico proiettato con un modello ARIMA(0,1,0)

$$k_t^{(2)} = k_{t-1}^{(2)} + \mu_{k^{(2)}} + \sigma_{k^{(2)}} e_t$$

- ▶ con errori  $e_t$  i.i.d. con distribuzione  $N(0,1)$

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# Il modello Poisson log bilineare

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- ▶ Il modello LC assume implicitamente che gli errori casuali siano omoschedastici → ipotesi poco realistica per età elevate → maggiore variabilità della mortalità causa un minor numero di esposti al rischio
- ▶ Proposta di Brouhns et al. nel 2002. Tassi centrali di mortalità modellizzati tramite il modello Lee-Carter

$$\log(m_{x,t}) = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)}$$

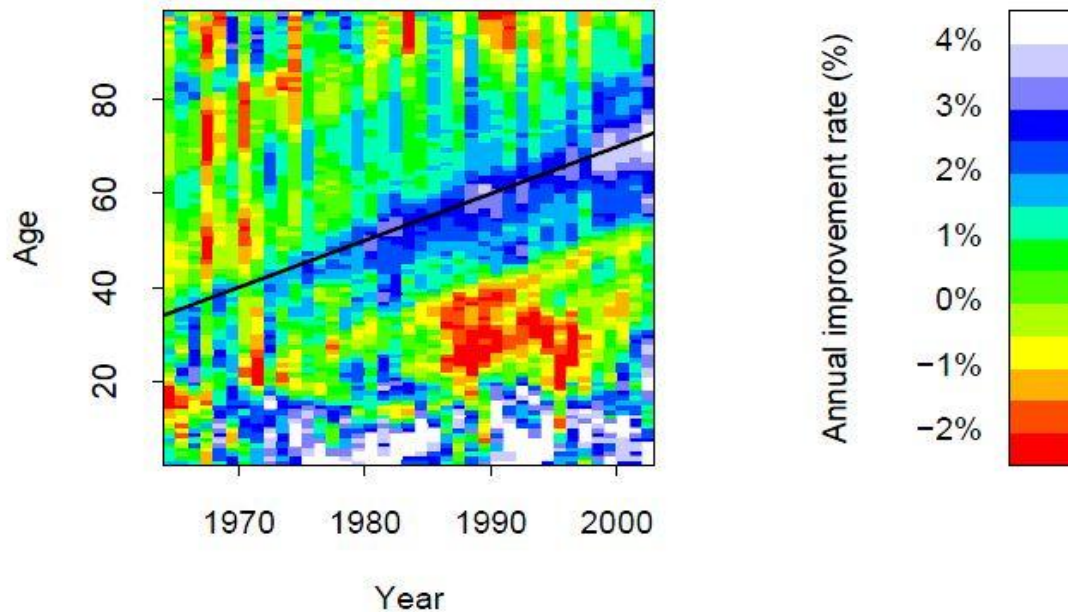
- ▶ Decessi distribuiti come una Poisson

$$D_{x,t} \sim \text{Poisson} \left( E_{x,t} m_{x,t} \right)$$

- ▶ Variazione rispetto al Lee-Carter originario: introduzione di una variazione casuale del numero di decessi di tipo Poisson al posto del termine di errore additivo  $\varepsilon_{x,t}$ . Ipotesi realistica per età elevate

# Effetto coorte

- ▶ In alcuni Paesi si osservano tassi di mortalità che sembrano influenzati non solo dall'età e dall'anno di calendario ma anche dall'anno di nascita della coorte.
- ▶ Per dare evidenza di questo effetto si possono analizzare i tassi di incremento annuo della mortalità



Fonte: Cairns et al. (2007) "A quantitative comparison of stochastic mortality models using data from England & Wales and the United States" WP0701, Pension Institute

# I modelli di proiezione della mortalità testati

- ▶ Appartengono ai principali modelli di proiezione della mortalità proposti in letteratura

Modello	Formula
Lee-Carter (LC)	$\log(m_{x,t}) = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)}$
Renshaw-Haberman (RH)	$\log(m_{x,t}) = \beta_x^{(1)} + \beta_x^{(2)} k_t^{(2)} + \beta_x^{(3)} \gamma_{t-x}^{(3)}$
Cairns-Blake-Dowd (CBD)	$\text{logit}(q_{x,t}) = k_t^{(1)} + k_t^{(2)}(x - \bar{x})$
Cairns-Blake-Dowd-2 (CBD-2)	$\text{logit}(q_{x,t}) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + \gamma_{t-x}^{(3)}$

- ▶ Funzione logit:  $\text{logit } q_{x,t} = \log\left(\frac{q_{x,t}}{p_{x,t}}\right)$  con  $q_{x,t} \approx 1 - e^{-m_{x,t}}$
- ▶  $\gamma_{t-x}^{(3)}$  fattore di coorte

$$\gamma_c^{(3)} = \gamma_{c-1}^{(3)} + \phi(\gamma_{c-1}^{(3)} - \gamma_{c-2}^{(3)}) + \mu_{\gamma^{(3)}} + \sigma_{\gamma^{(3)}} e_c$$

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# I criteri di scelta del modello

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- ▶ Cairns et al. (2008) suggeriscono alcuni criteri per scegliere tra i vari modelli di proiezione della mortalità. Ad esempio:
  - Modello coerente con i dati storici
  - Dinamiche future a lungo termine del modello biologicamente ragionevoli
  - Stime dei parametri robuste rispetto al periodo di dati e intervalli di età impiegati
  - Previsioni del modello robuste rispetto al periodo di dati e intervalli di età impiegati
  - Livelli di previsione dell'incertezza e traiettorie centrali plausibili e coerenti con le tendenze storiche e la variabilità dei dati di mortalità



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# I criteri di scelta del modello

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- Modello semplice da attuare mediante metodi analitici o veloci algoritmi numerici
- Modello relativamente parsimonioso
- Modello utilizzabile per generare percorsi campione e calcolare intervalli di previsione
- Modello che consente di integrare l'incertezza del parametro nelle simulazioni
- Almeno per alcuni Paesi, modello che incorpora un effetto stocastico di coorte

# Criterio di scelta del modello

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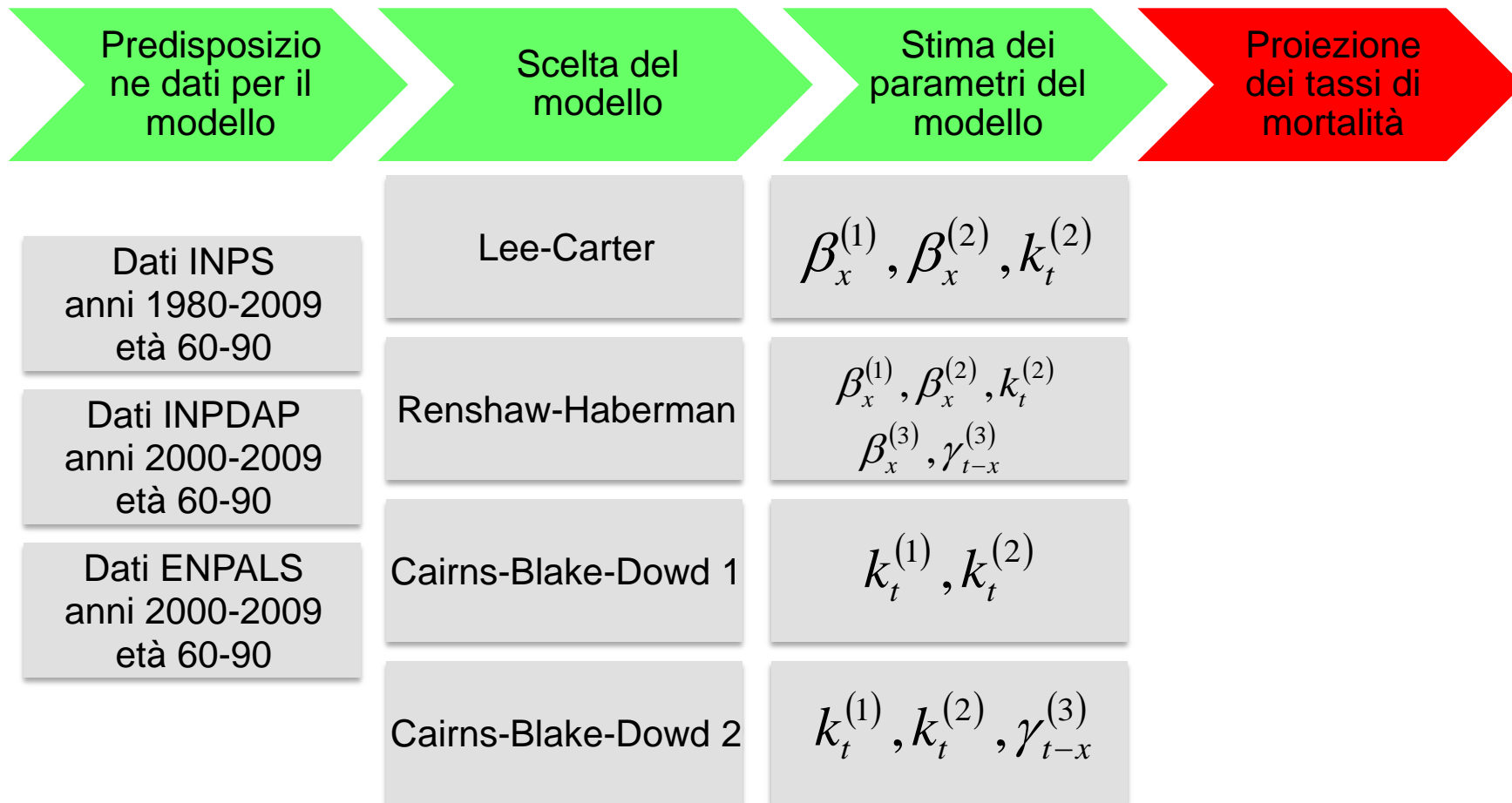
- ▶ Bayes Information Criterion (BIC): criterio obiettivo di scelta del modello basato sulla qualità statistica del fit

$$BIC = l(\hat{\rho}) - 0.5K \ln(N)$$

- $\ln(N)$  è la funzione di log-verosimiglianza
- $\rho$  insieme dei parametri da stimare con la funzione di verosimiglianza
- $\hat{\rho}$  stima di massima verosimiglianza del vettore dei parametri
- $N$  vettore del numero delle osservazioni
- $K$  numero effettivo dei parametri stimati
- Funzione di log-verosimiglianza del modello:

$$l(\rho; D; E) = \sum_{x,t} \left\{ D_{x,t} \ln[E_{x,t} m_{x,t}(\rho)] - E_{x,t} m_{x,t}(\rho) - \ln(D_{x,t}!) \right\}$$

# Le fasi del progetto



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# Dati INPS

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## ▶ Lavoratori dipendenti

- Categoria AEFG:
  - dipendenti (cat: A)
  - trasporti (cat: E)
  - telefonici (cat: F)
  - elettrici (cat: G)

Età: 60-95

Periodo: 1980-2009

## ▶ Lavoratori non dipendenti

- Categoria B (CDCM)
- Categoria C (artigiani e commercianti)

## ▶ Totale

# Risultati della procedura di fitting

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## INPS, dip, maschi, 1980-2009, 60-95

Modello	BIC	MLL	Rank(BIC)	Rank(MLL)
LC	-6539	-6190	3	3
RH	-6238	-5545	1	1
CBD1	-6670	-6461	4	4
CBD2	-6351	-5923	2	2

## INPS, dip, femmine, 1980-2009, 60-95

Modello	BIC	MLL	Rank(BIC)	Rank(MLL)
LC	-5782	-5433	1	2
RH	-5948	-5255	2	1
CBD1	-7042	-6833	4	4
CBD2	-6135	-5707	3	3

## INPS, aut, maschi, 1980-2009, 60-95

Modello	BIC	MLL	Rank(BIC)	Rank(MLL)
LC	-5444	-5096	2	2
RH	-5433	-4739	1	1
CBD1	-5670	-5461	4	4
CBD2	-5572	-5144	3	3

## INPS, aut, femmine, 1980-2009, 60-95

Modello	BIC	MLL	Rank(BIC)	Rank(MLL)
LC	-5100	-4752	1	2
RH	-5328	-4634	3	1
CBD1	-5881	-5672	4	4
CBD2	-5222	-4794	2	3

## INPS, tot, maschi, 1980-2009, 60-95

Modello	BIC	MLL	Rank(BIC)	Rank(MLL)
LC	-6739	-6391	3	3
RH	-6364	-5671	1	1
CBD1	-6835	-6626	4	4
CBD2	-6446	-6017	2	2

## INPS, tot, femmine, 1980-2009, 60-95

Modello	BIC	MLL	Rank(BIC)	Rank(MLL)
LC	-5939	-5591	1	2
RH	-6145	-5452	2	1
CBD1	-7817	-7608	4	4
CBD2	-6377	-5948	3	3

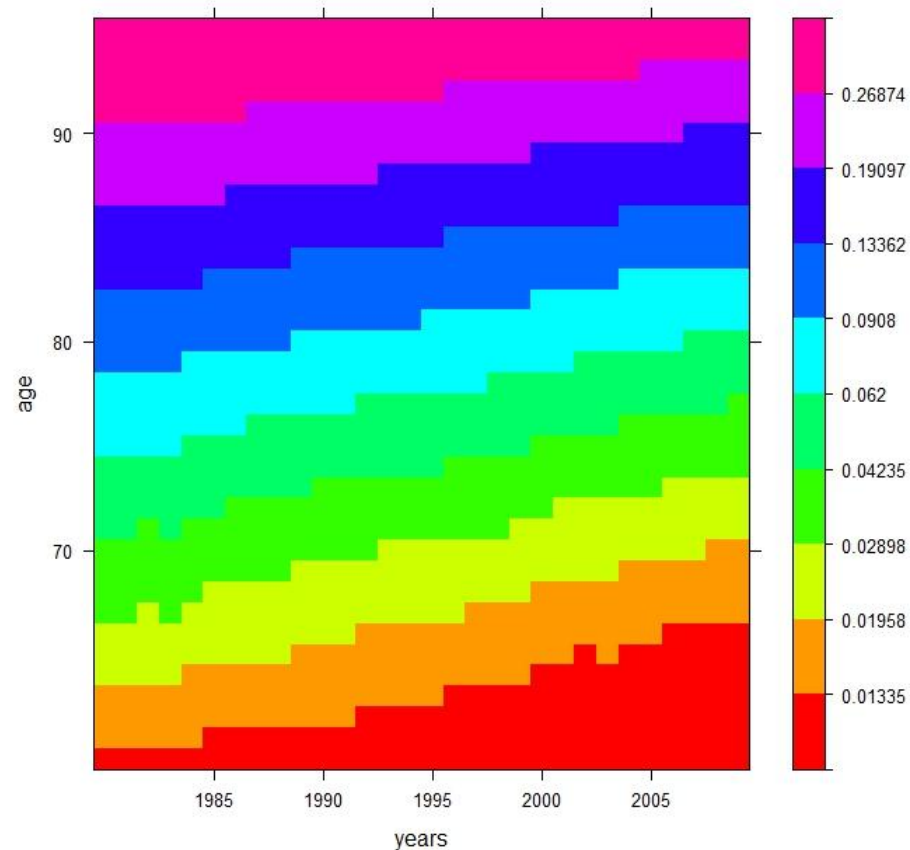
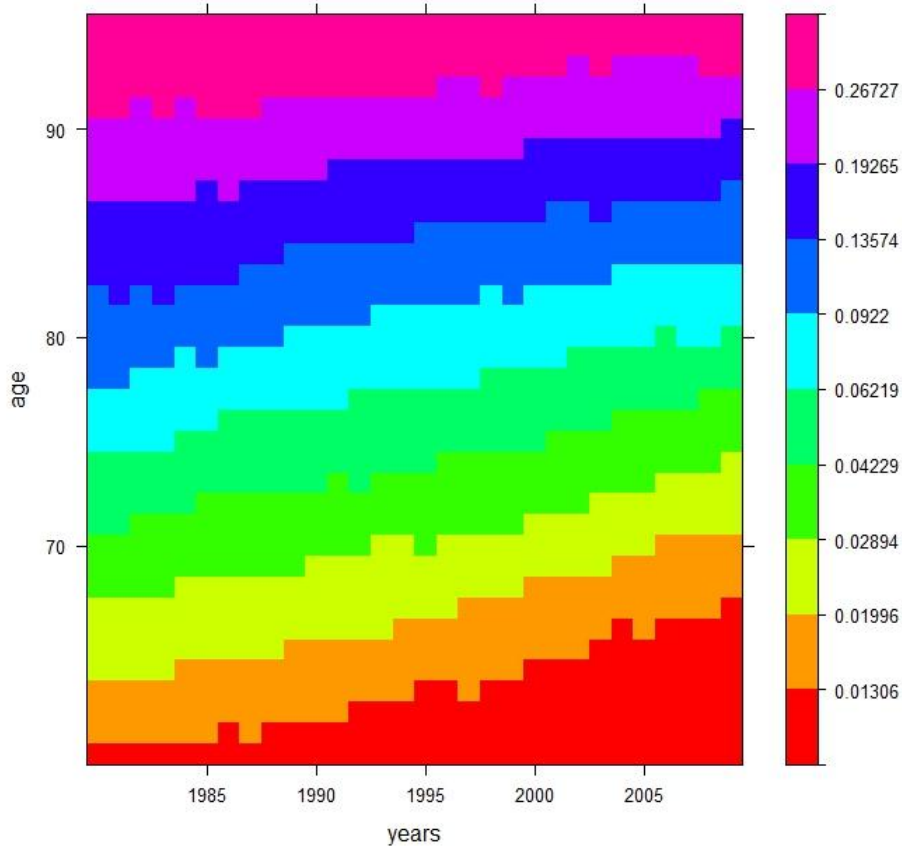
# **INPS: lavoratori dipendenti**

# Tassi centrali di mortalità: **Inps dip.**

Maschi

$m_{x,t}$  – dati **grezzi**

$m_{x,t}$  – dati fittati con il modello **Lee-Carter**

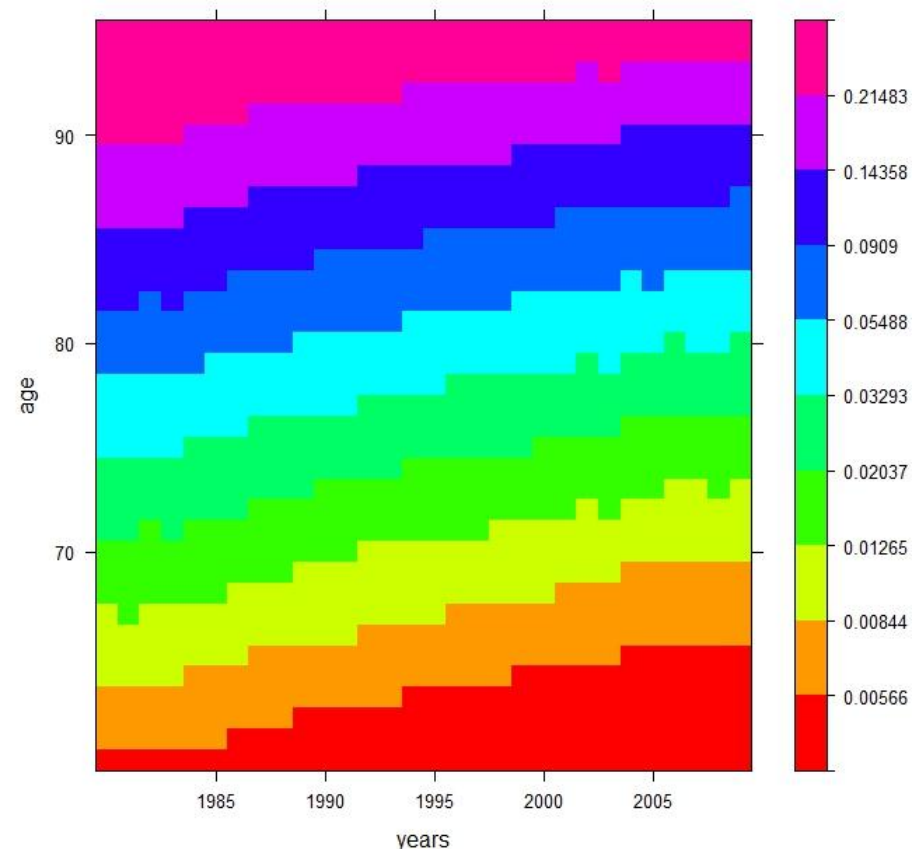
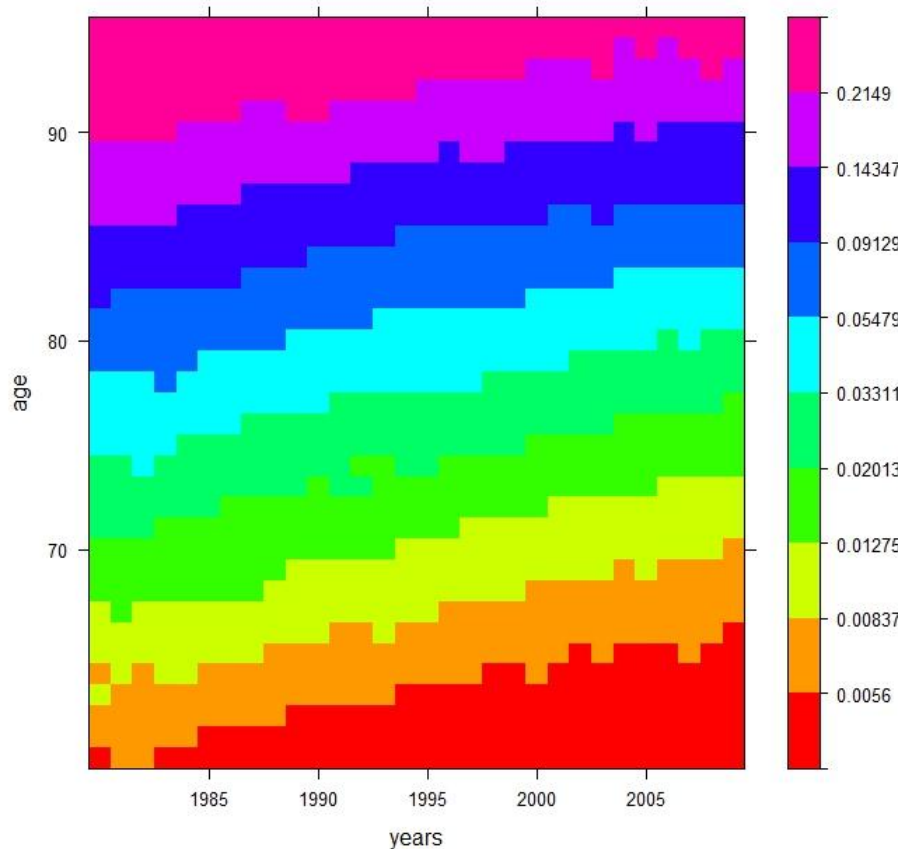


# Tassi centrali di mortalità: **Inps dip.**

Femmine

$m_{x,t}$  – dati **grezzi**

$m_{x,t}$  – dati fittati con il modello **Lee-Carter**





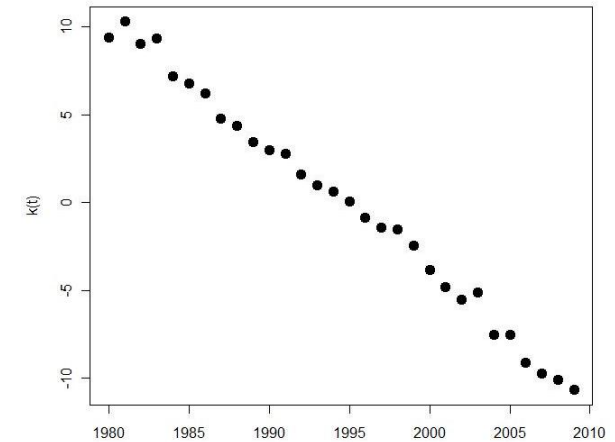
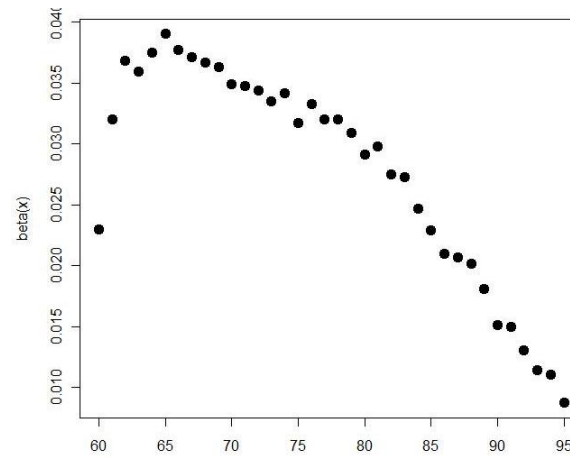
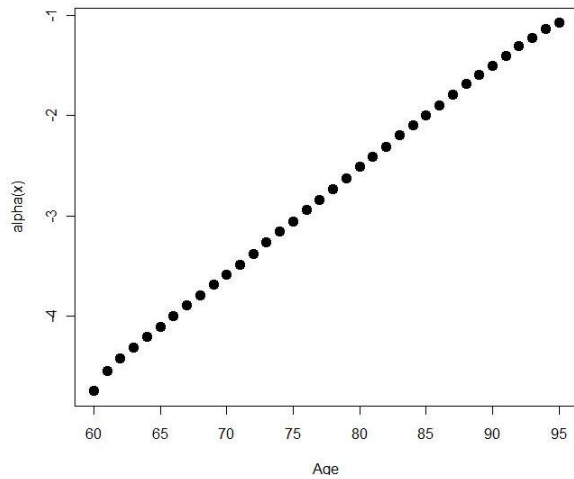
# Parametri del modello Lee-Carter: **Inps dip.**

Maschi

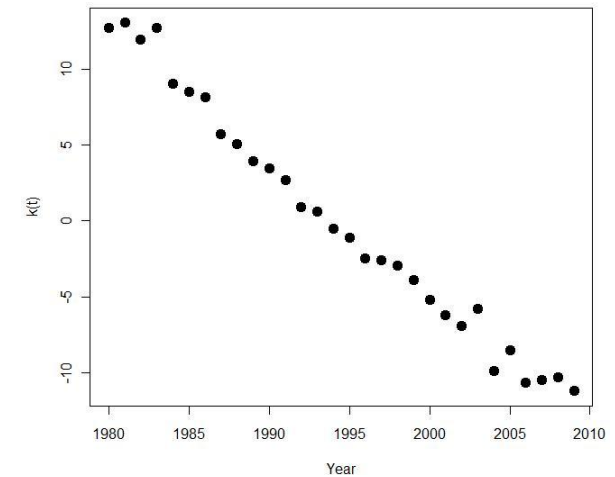
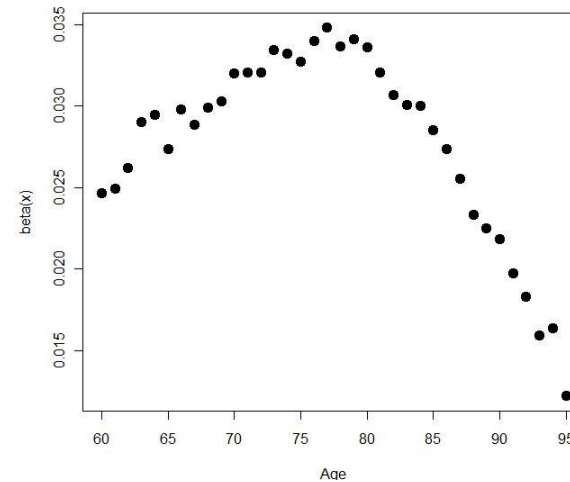
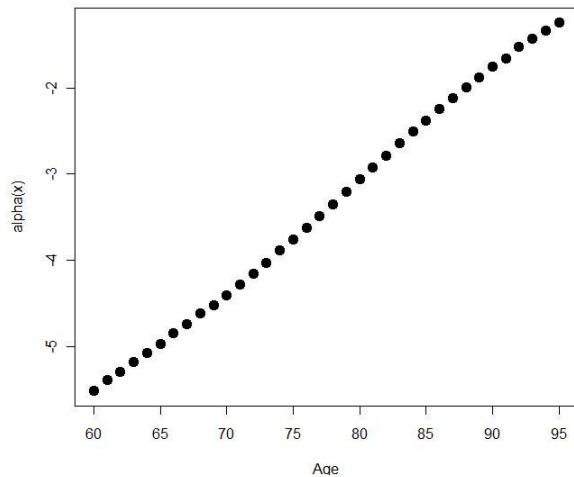
Beta 1

Beta 2

Kappa 2



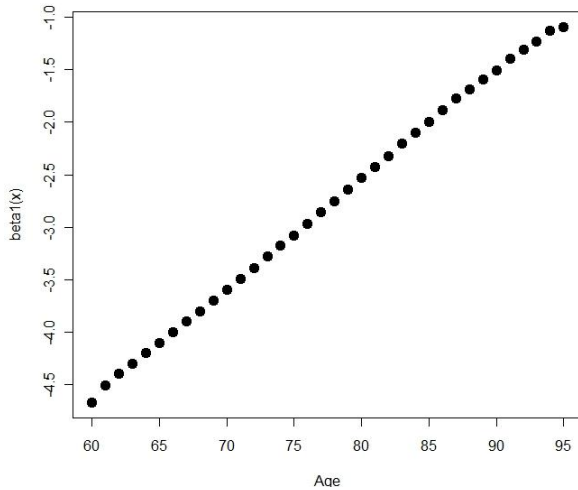
Femmine



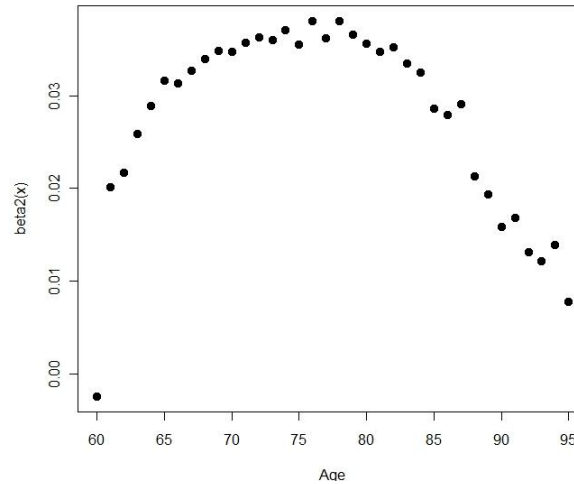
# Parametri del modello RH: **Inps dip.**

Maschi

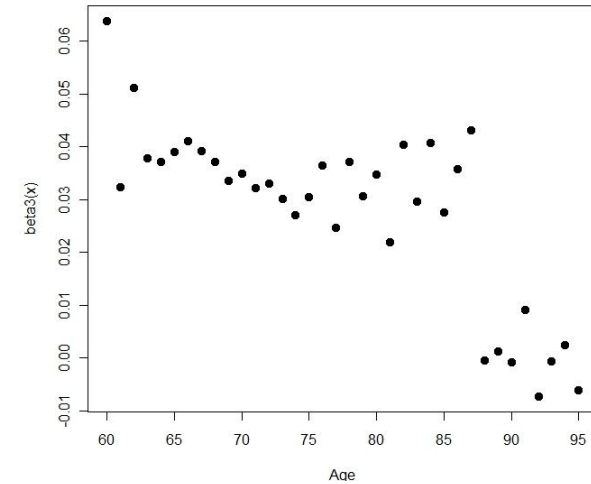
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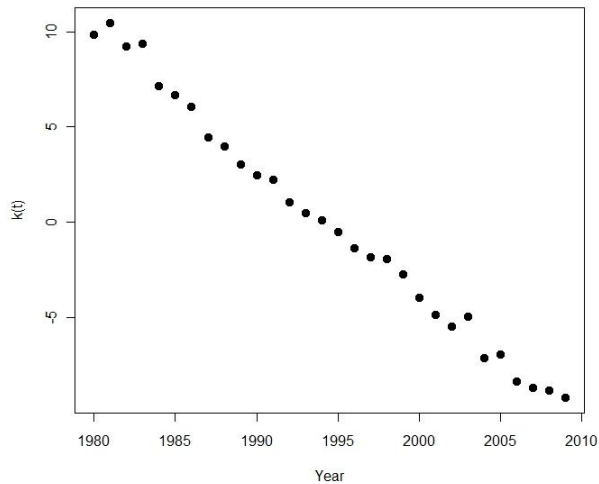
Beta2



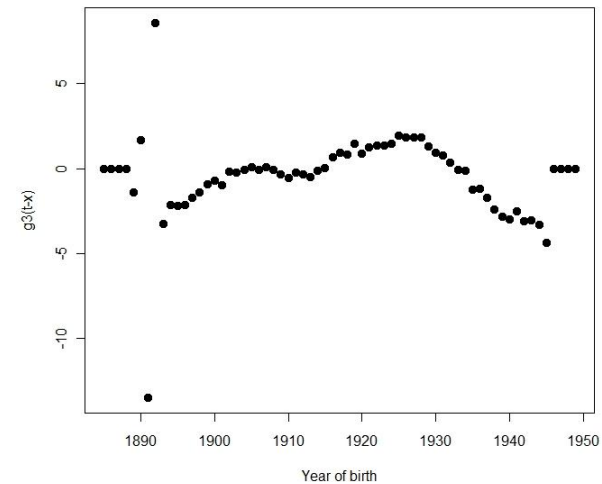
Beta3



Kappa2



g3(x)

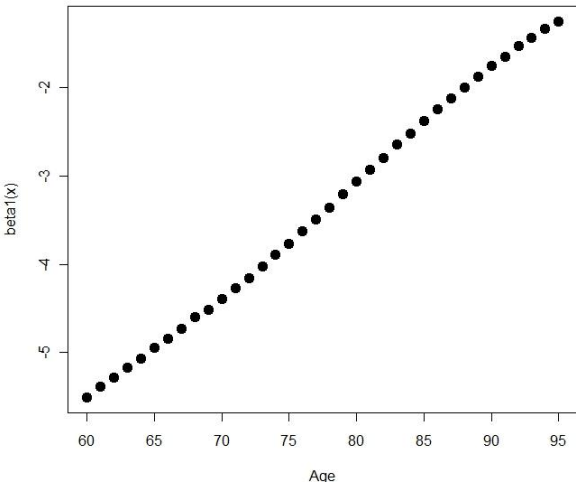


Gamma3

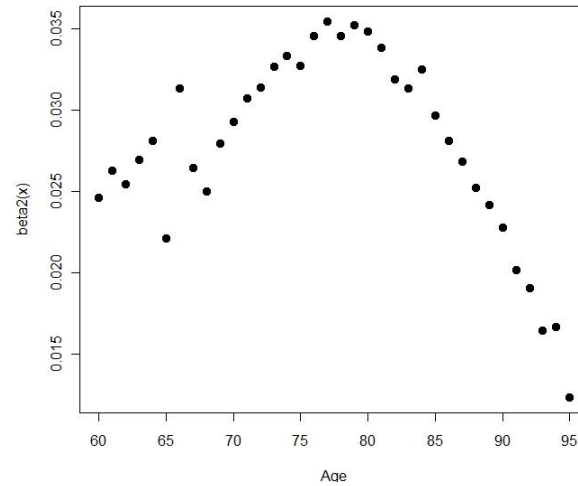
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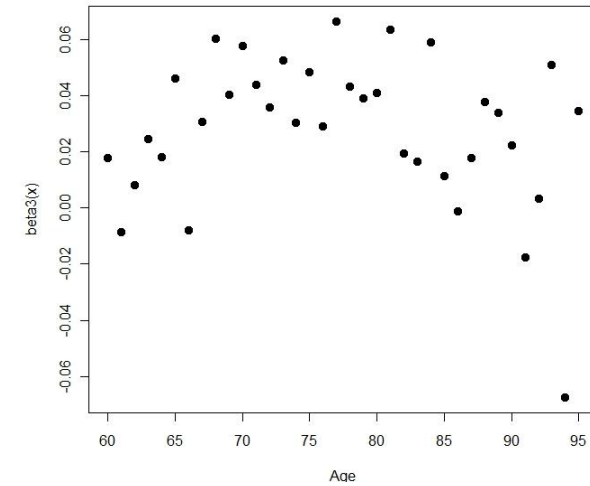
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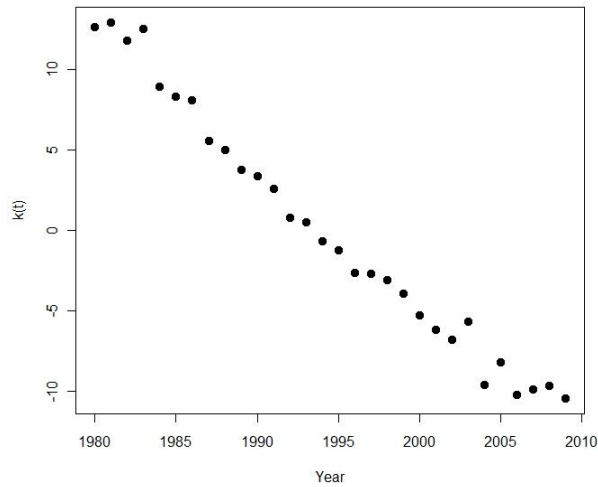
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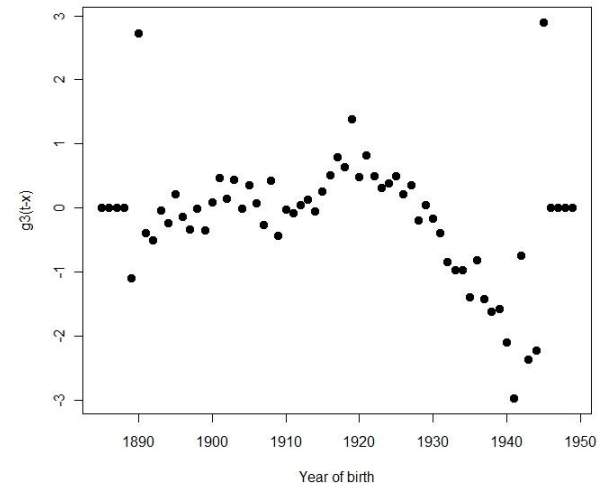
Beta3



Kappa2



Gamma3



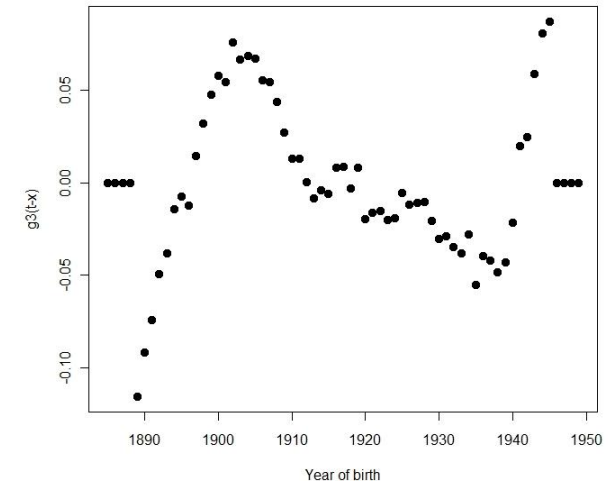
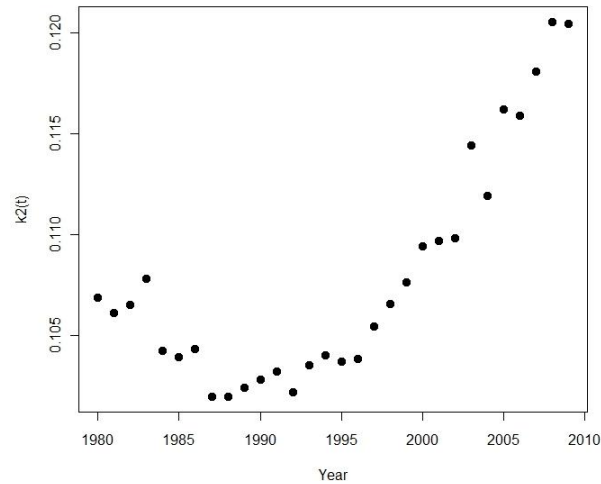
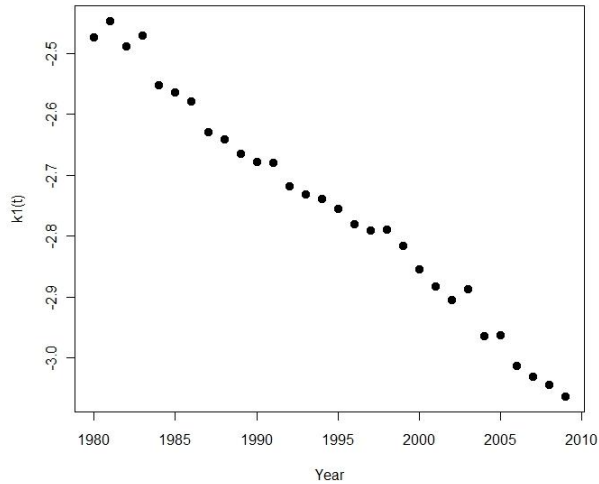
# Parametri del modello CBD-2: **Inps dip.**

Maschi

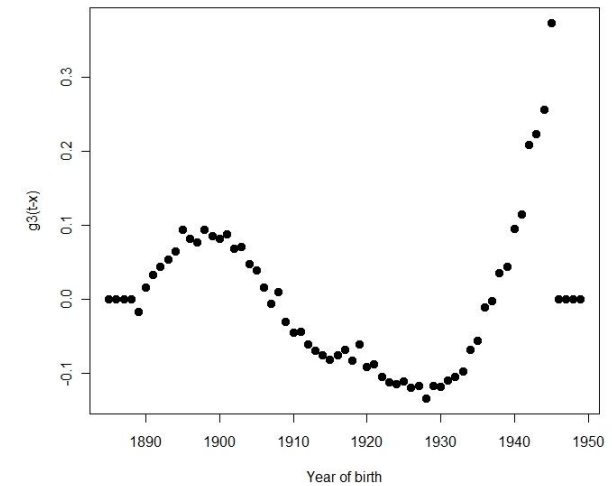
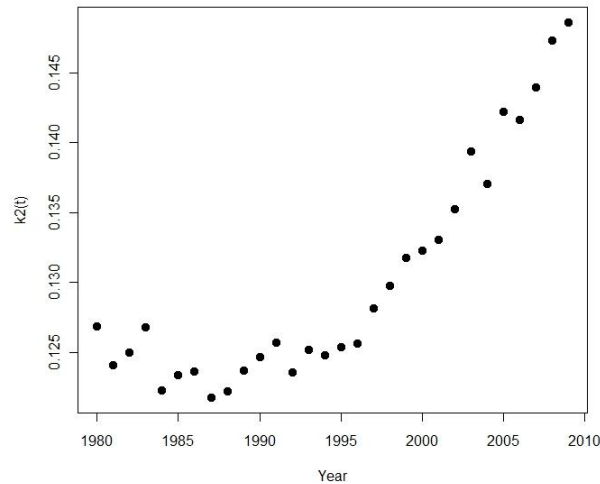
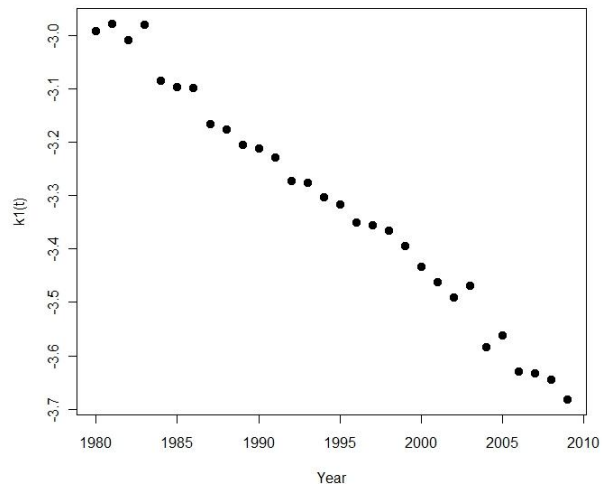
Kappa1

Kappa2

Gamma3



Femmine

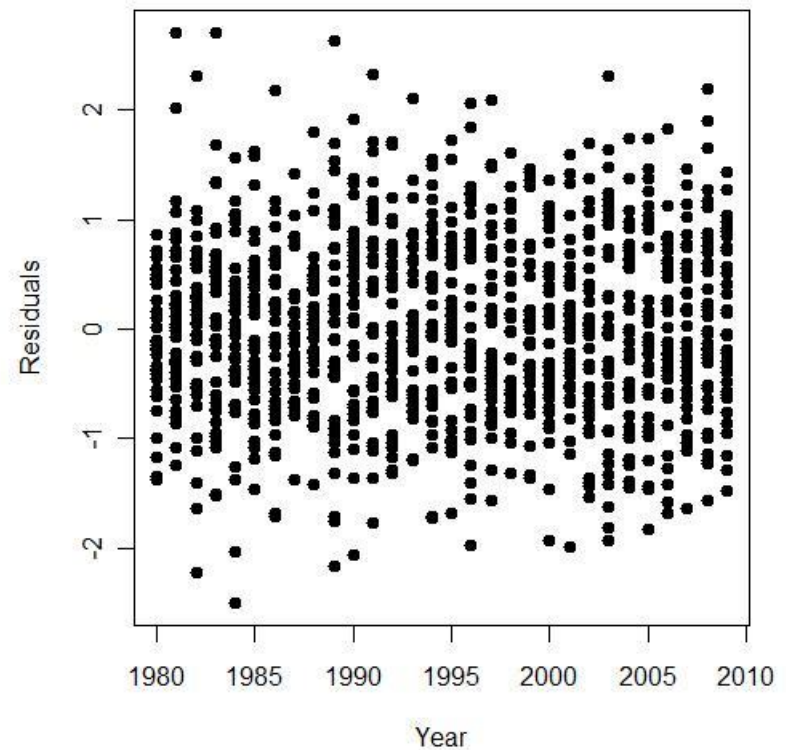
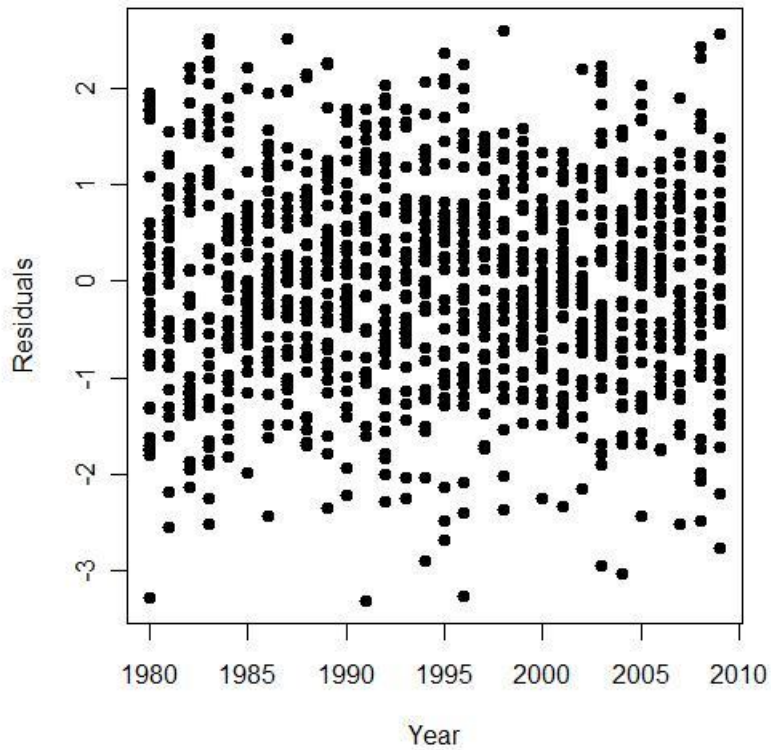


# Modello Lee-Carter: residui standardizzati

Maschi

INPS dipendenti

Femmine

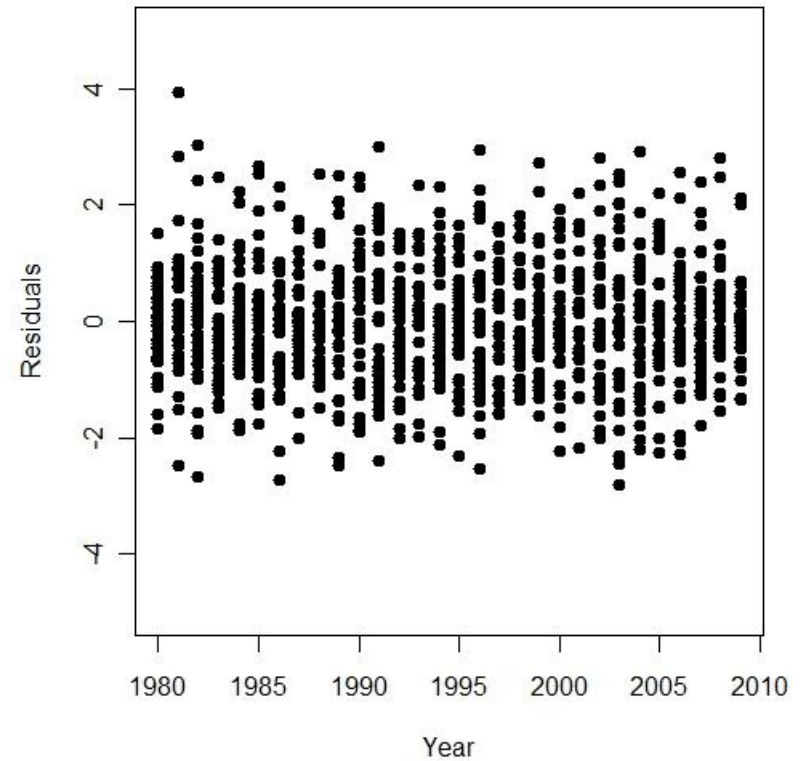
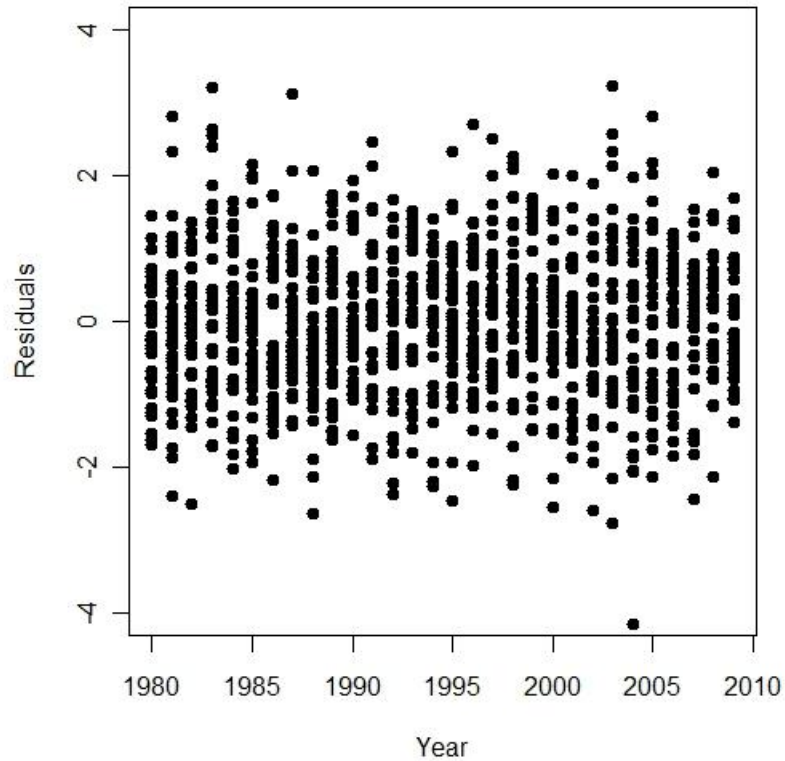


# Modello RH: residui standardizzati

Maschi

INPS dipendenti

Femmine

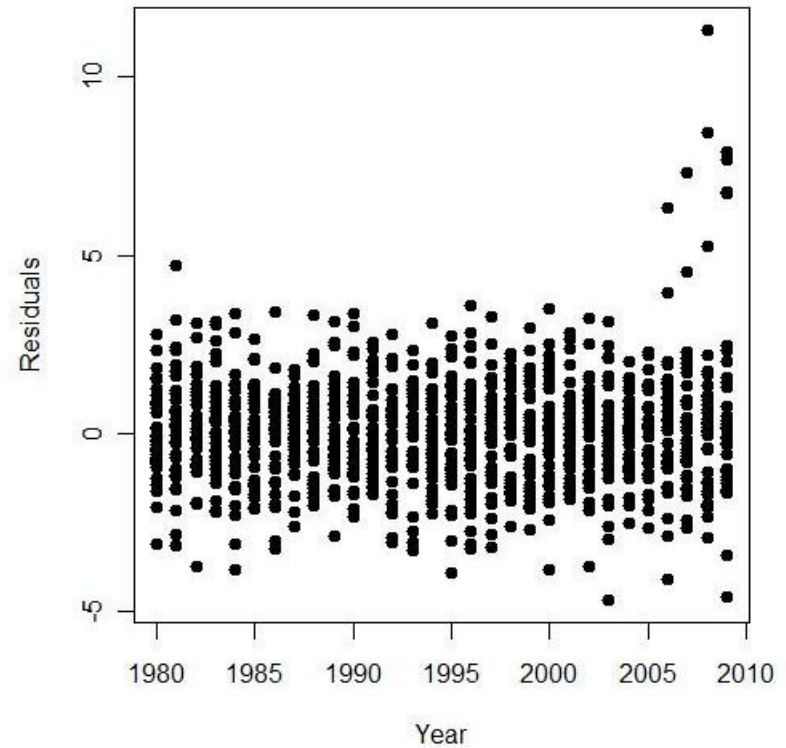
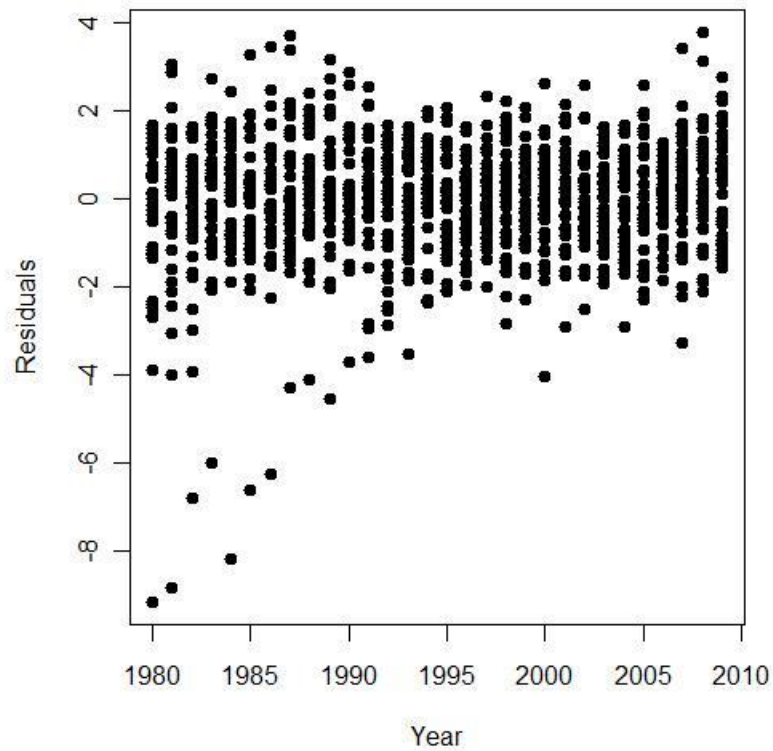


# Modello CBD-2: residui standardizzati

Maschi

INPS dipendenti

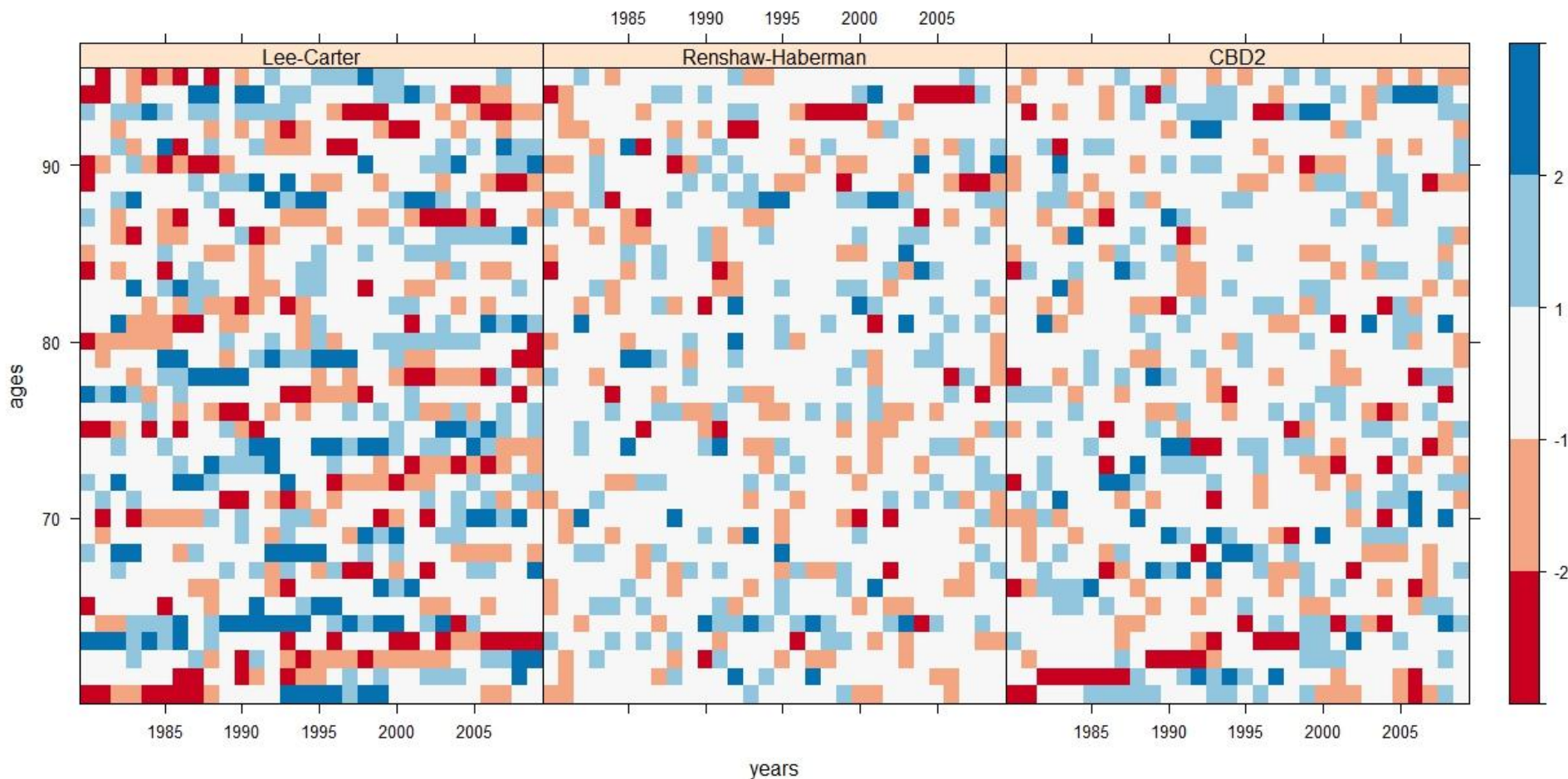
Femmine



# Confronto tra i modelli: residui standardizzati

Maschi

INPS dipendenti

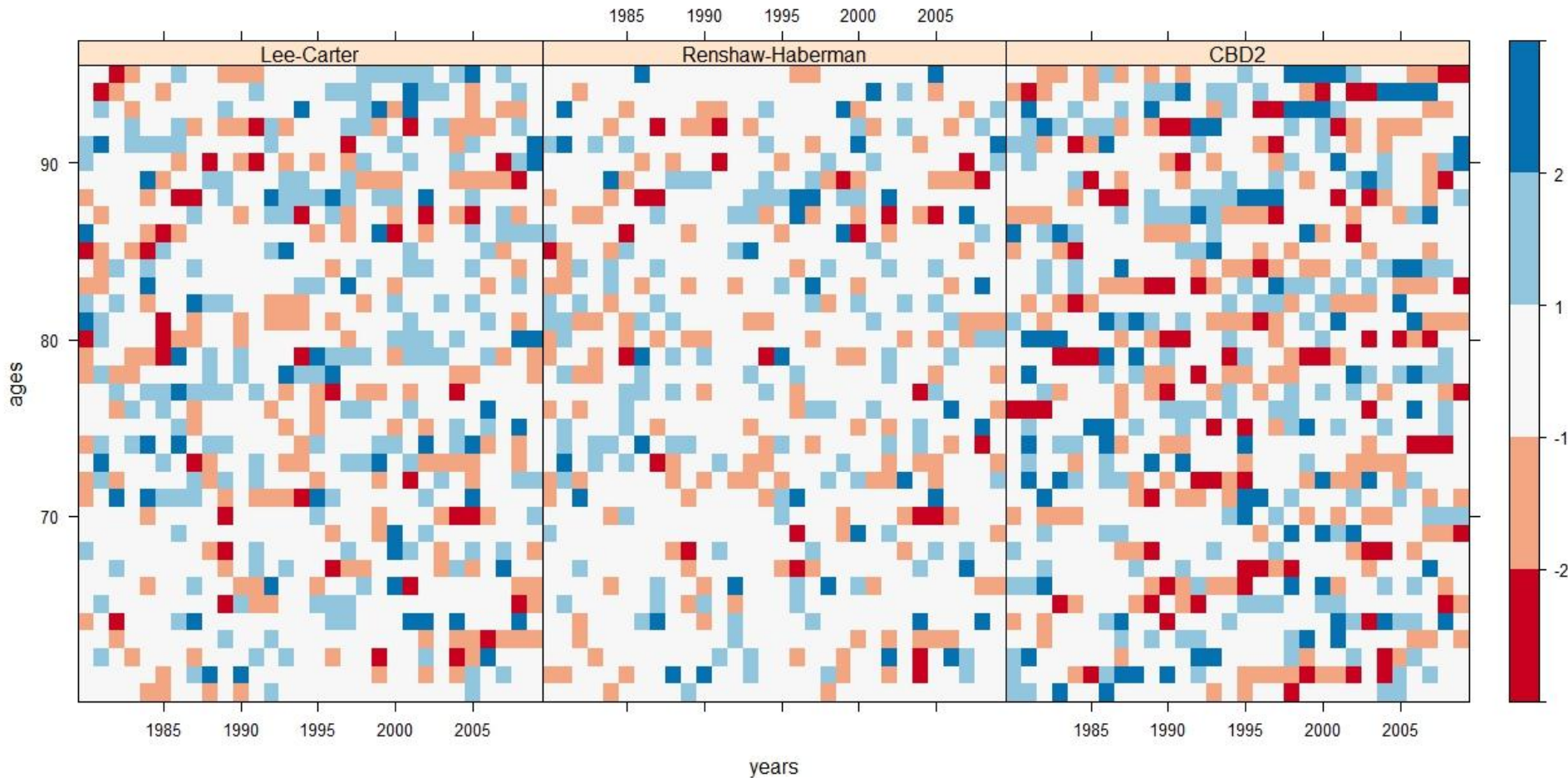




# Confronto tra i modelli: residui standardizzati

Femmine

INPS dipendenti

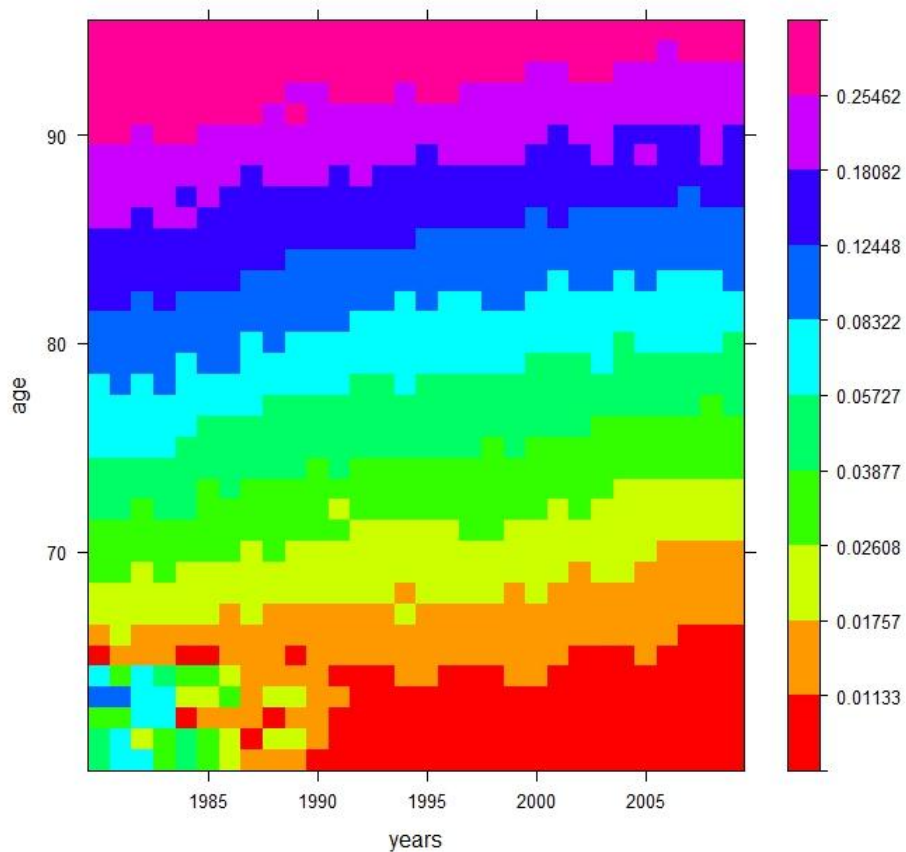


# INPS: lavoratori autonomi

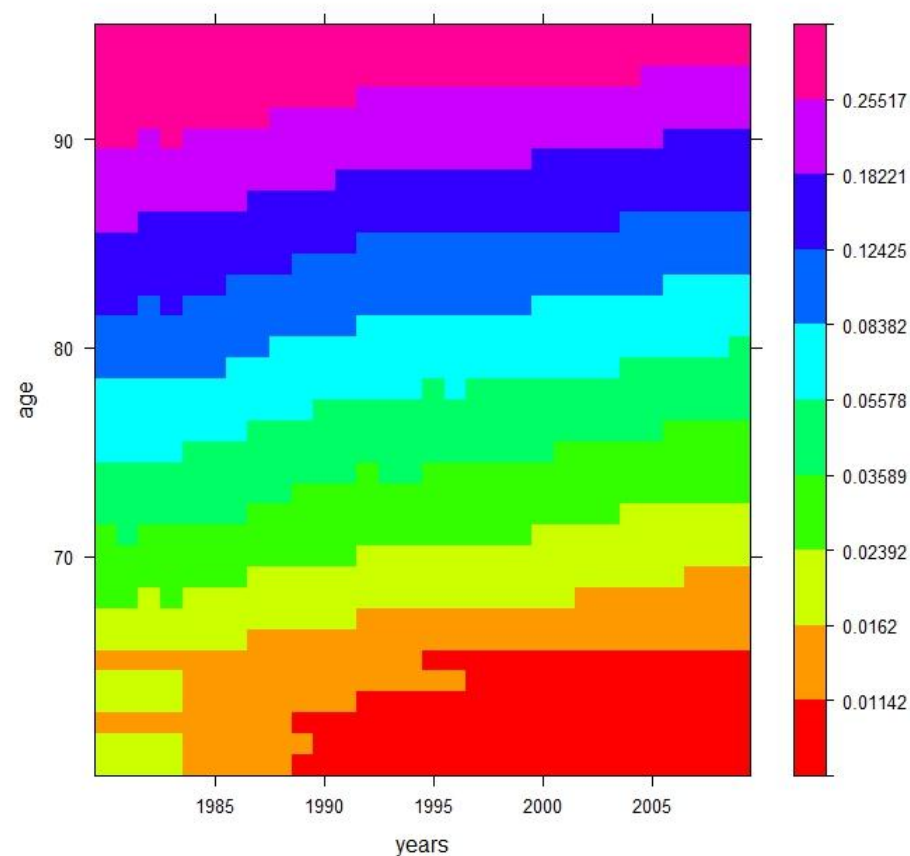
# Tassi centrali di mortalità: **Inps aut.**

Maschi

$m_{x,t}$  – dati **grezzi**



$m_{x,t}$  – dati fittati con il modello **Lee-Carter**

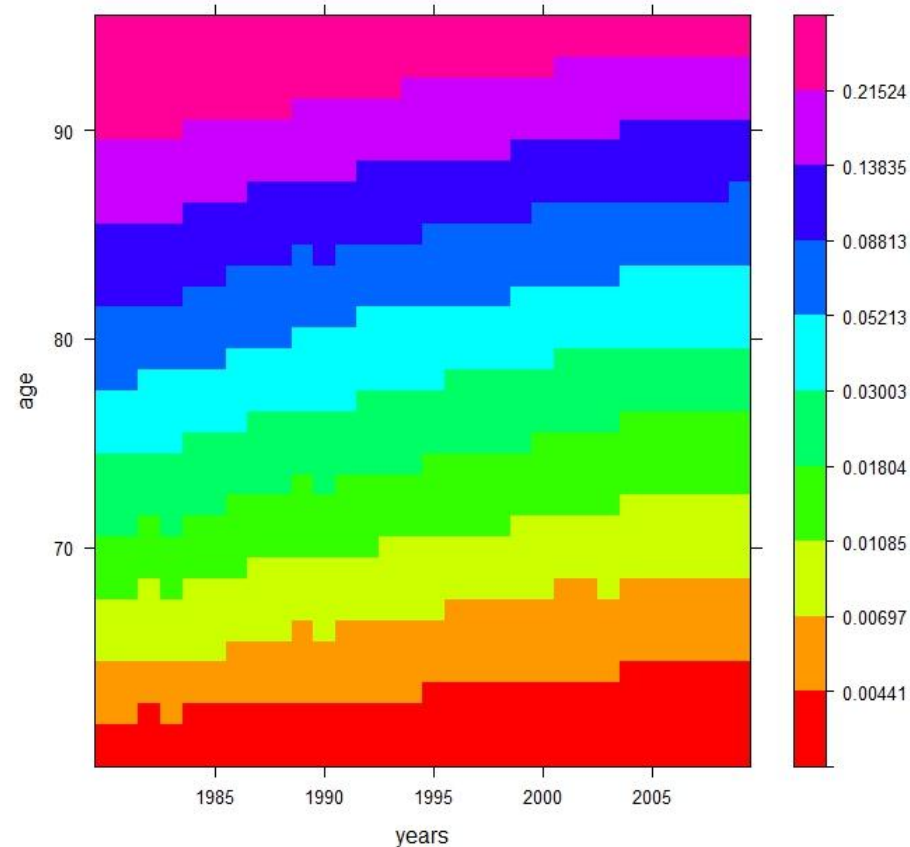
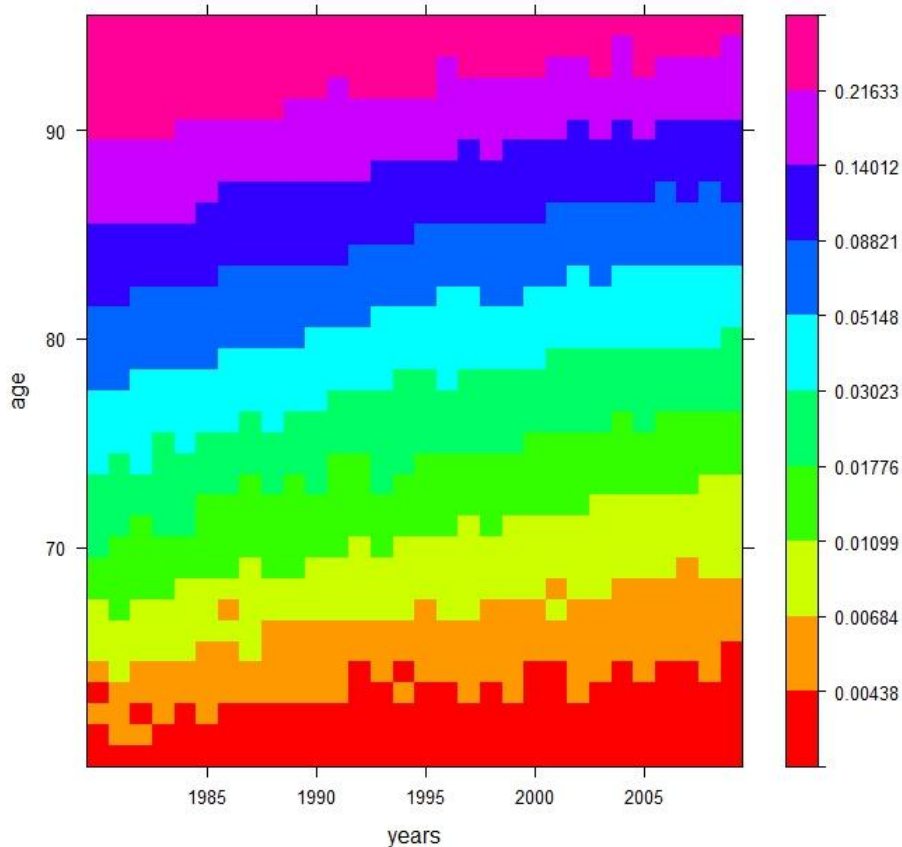


# Tassi centrali di mortalità: **Inps aut.**

Femmine

$m_{x,t}$  – dati **grezzi**

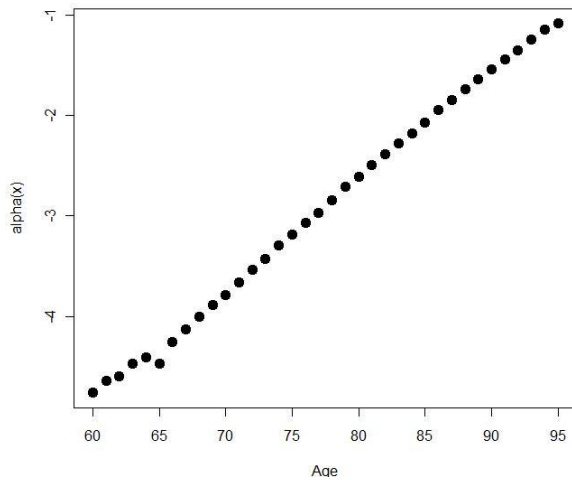
$m_{x,t}$  – dati fittati con il modello **Lee-Carter**



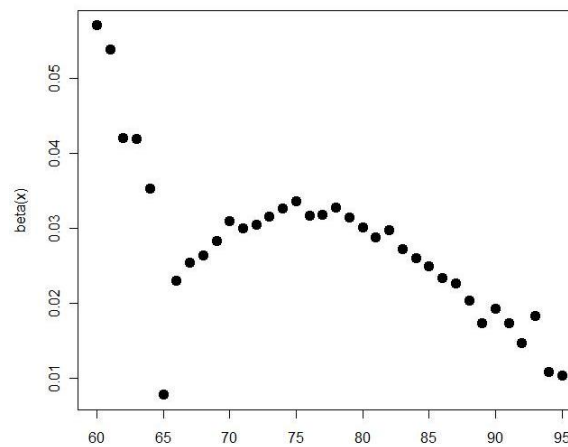
# Parametri del modello Lee-Carter: **Inps aut.**

Maschi

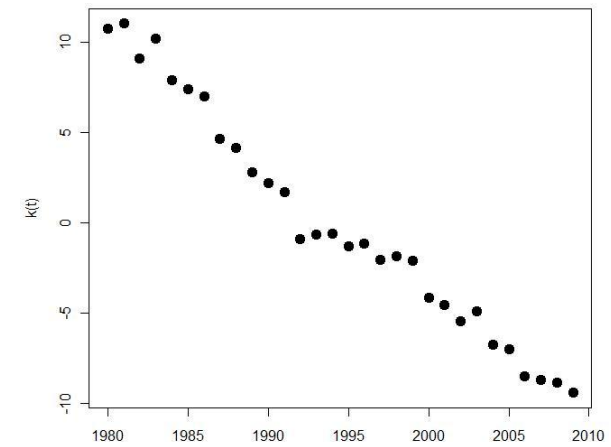
Beta 1



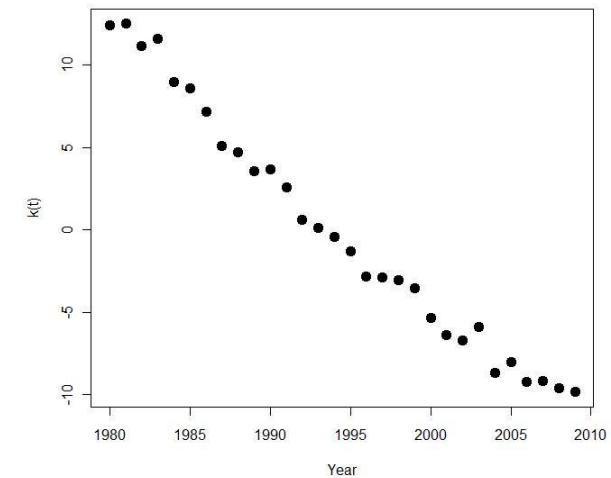
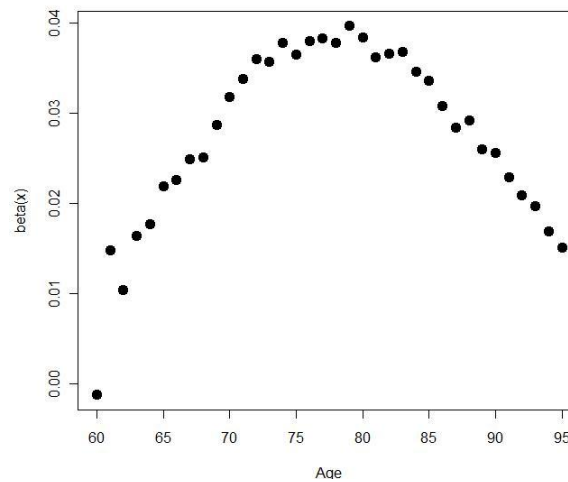
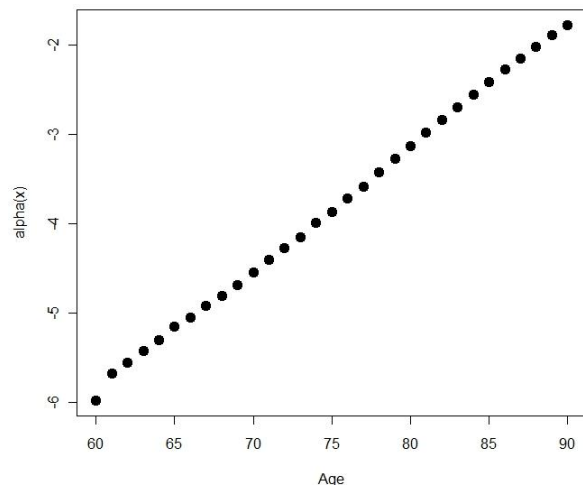
Beta 2



Kappa 2



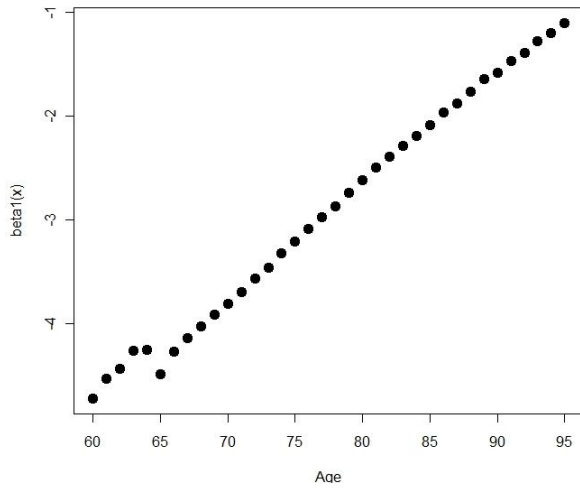
Femmine



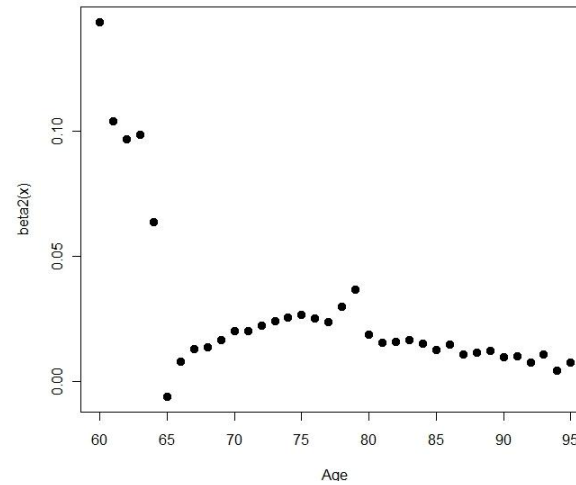
# Parametri del modello RH: **Inps aut.**

Maschi

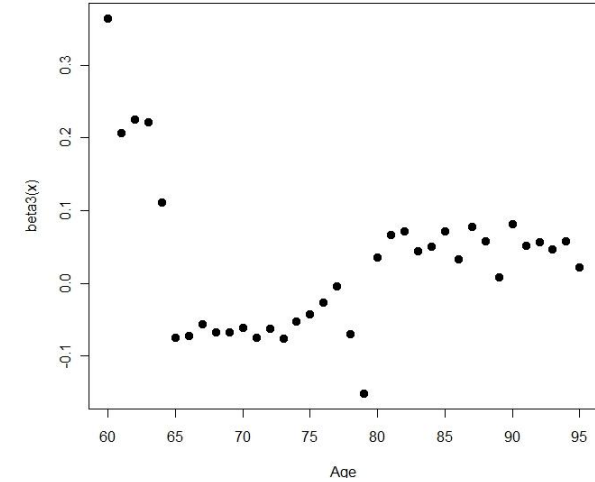
Beta1



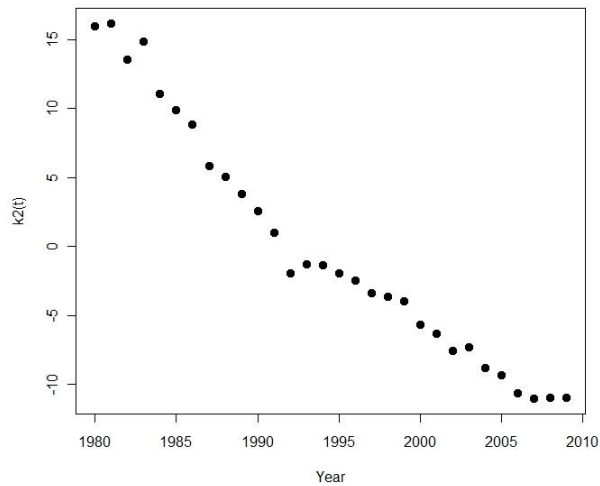
Beta2



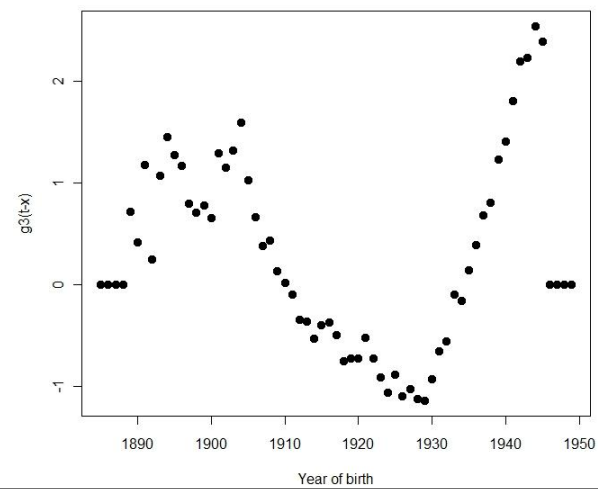
Beta3



Kappa2



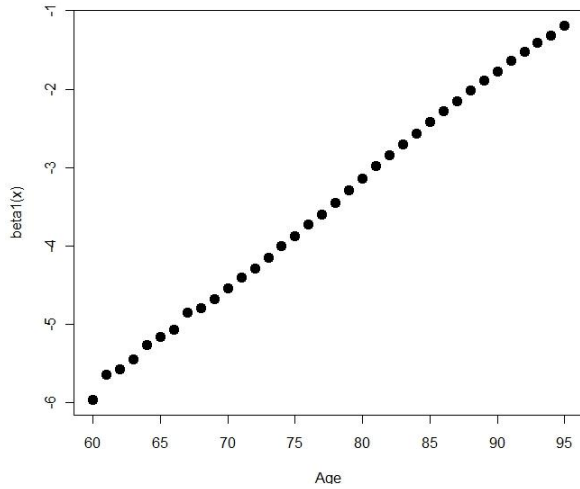
Gamma3



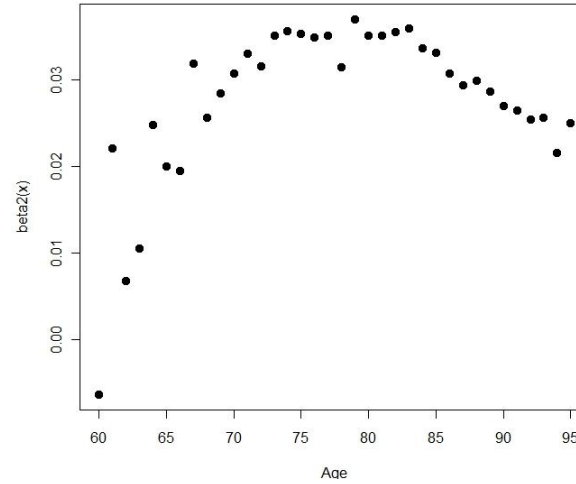
# Parametri del modello RH: **Inps aut.**

Femmine

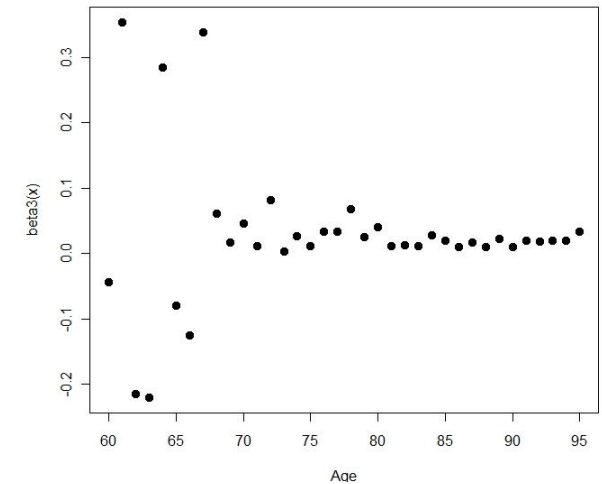
Beta1



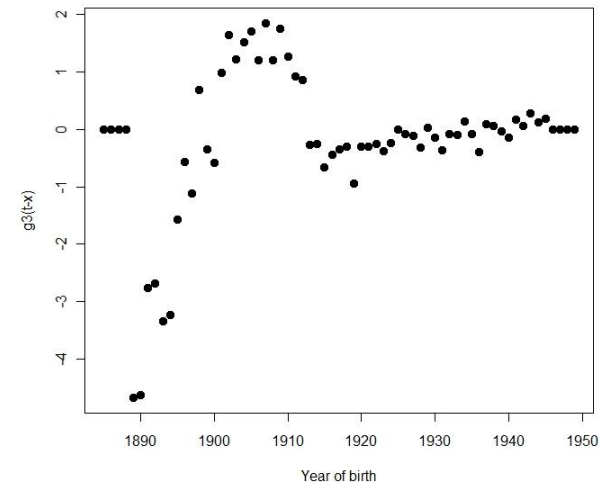
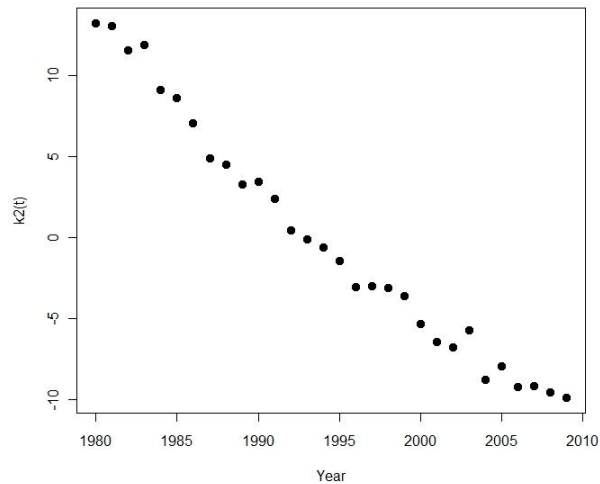
Beta2



Beta3



Kappa2



Gamma3

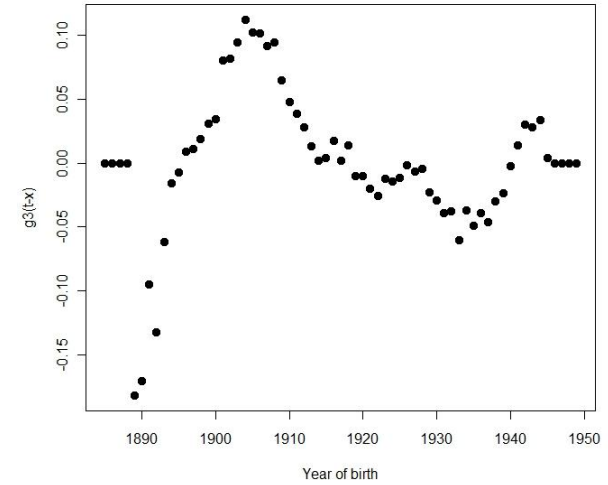
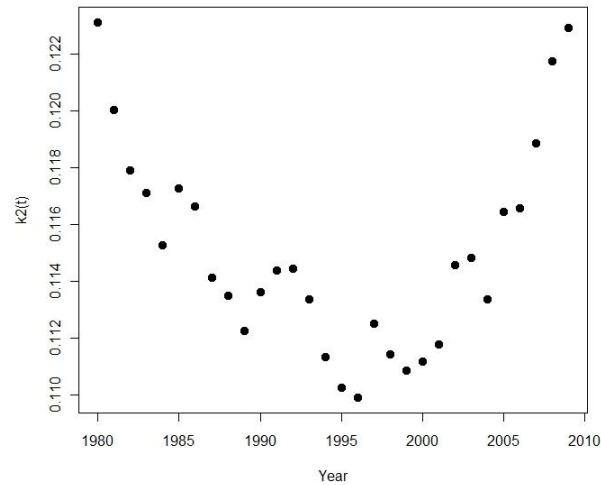
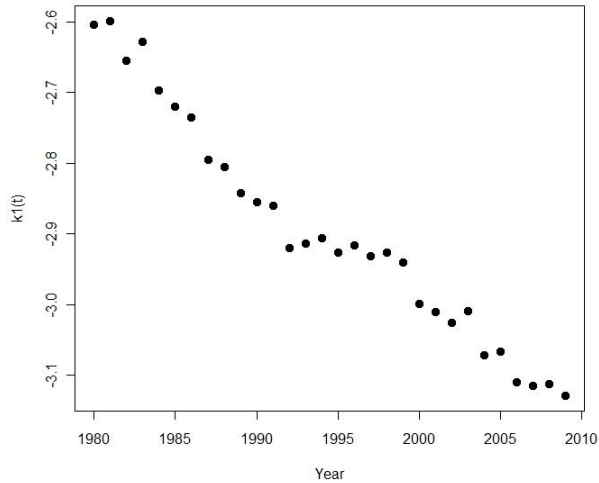
# Parametri del modello CBD-2: **Inps aut.**

Maschi

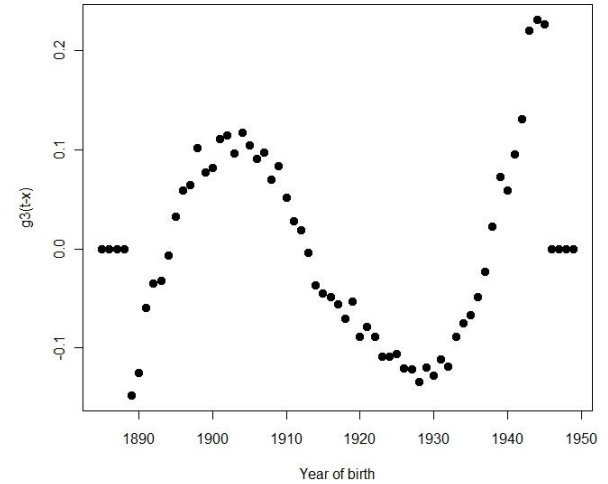
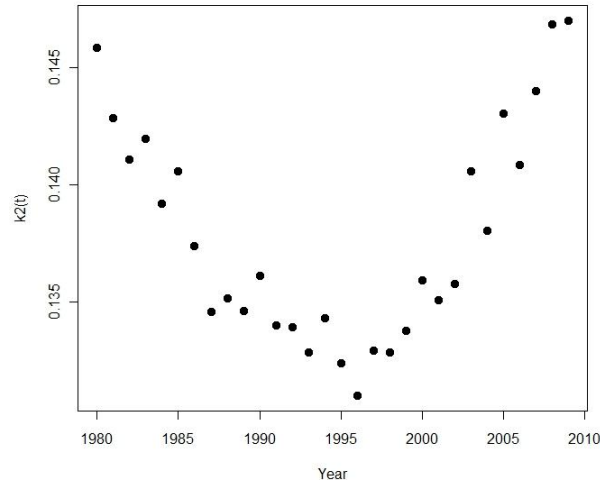
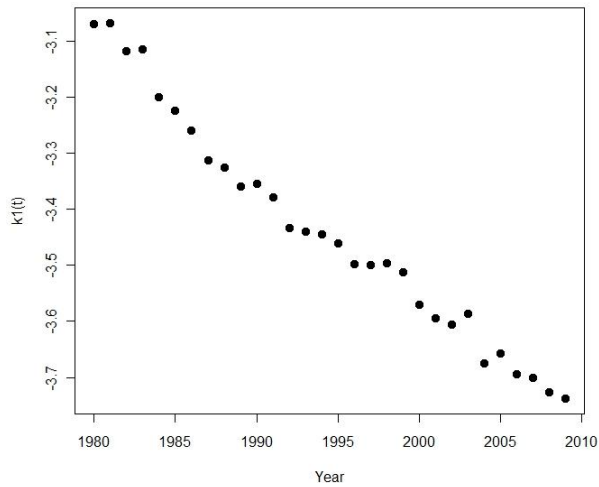
Kappa1

Kappa2

Gamma3



Femmine



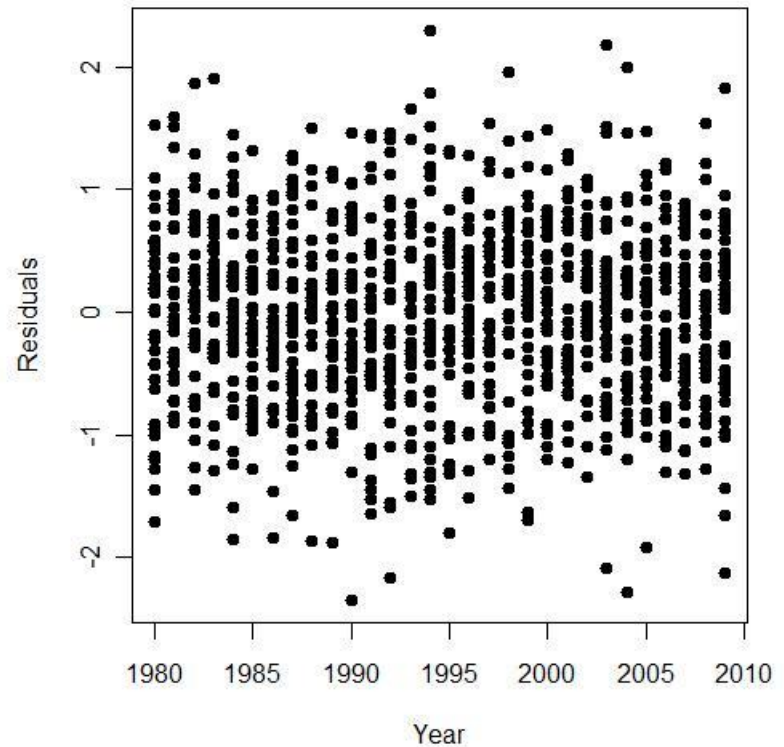
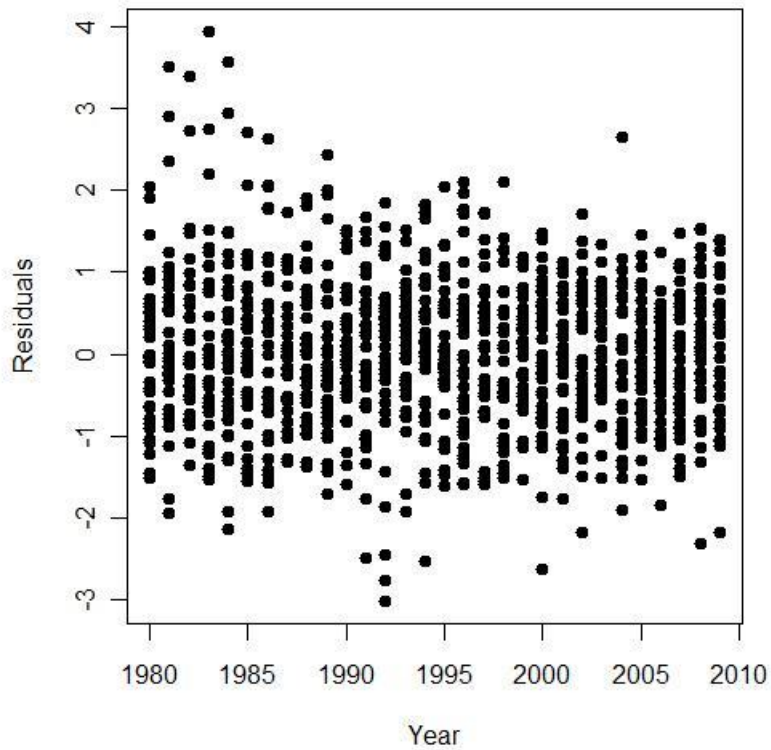


# Modello Lee-Carter: residui standardizzati

Maschi

INPS autonomi

Femmine

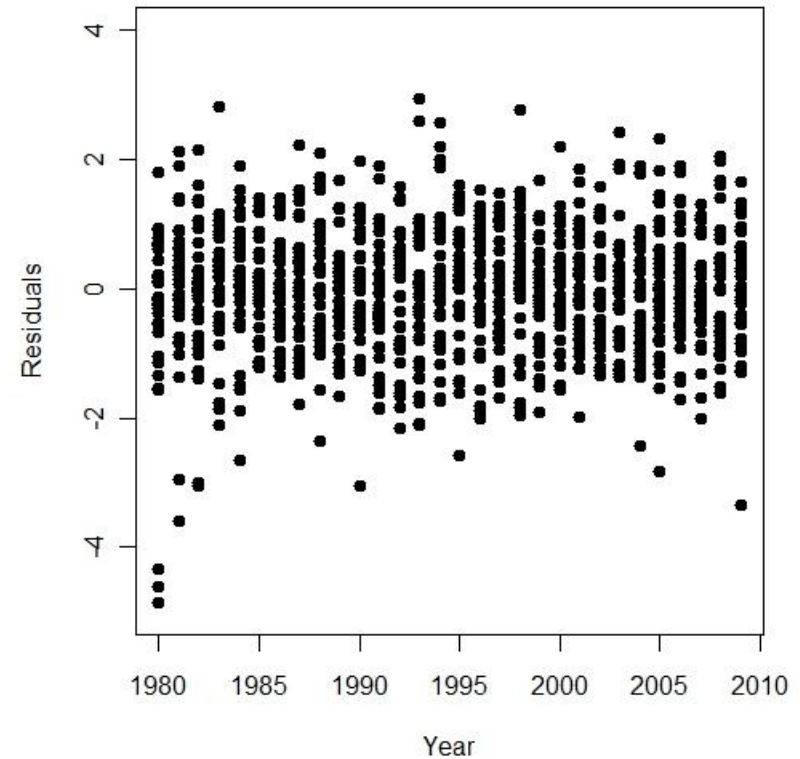
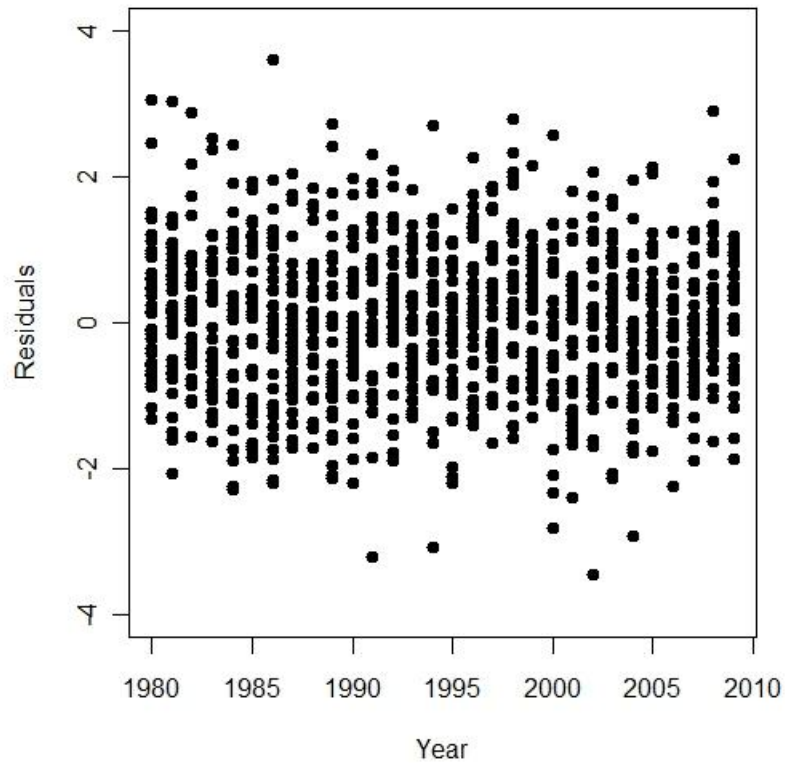


# Modello RH: residui standardizzati

Maschi

INPS autonomi

Femmine

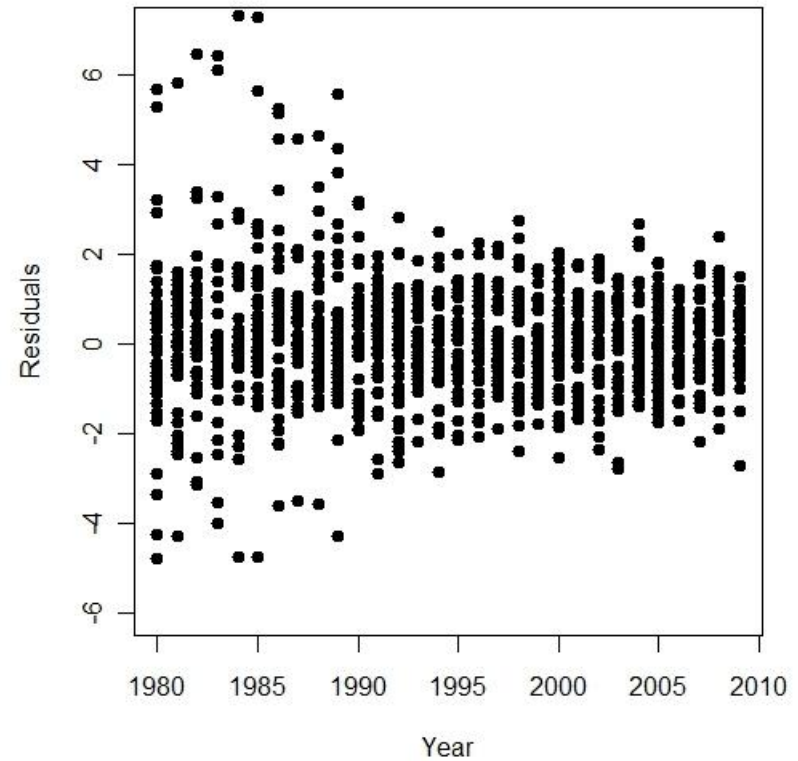
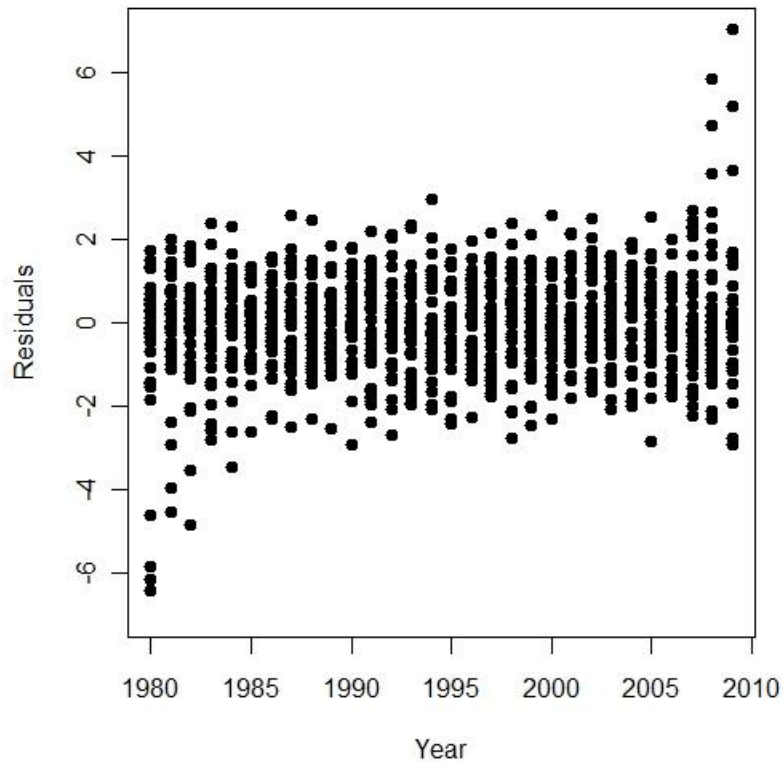


# Modello CBD-2: residui standardizzati

Maschi

INPS autonomi

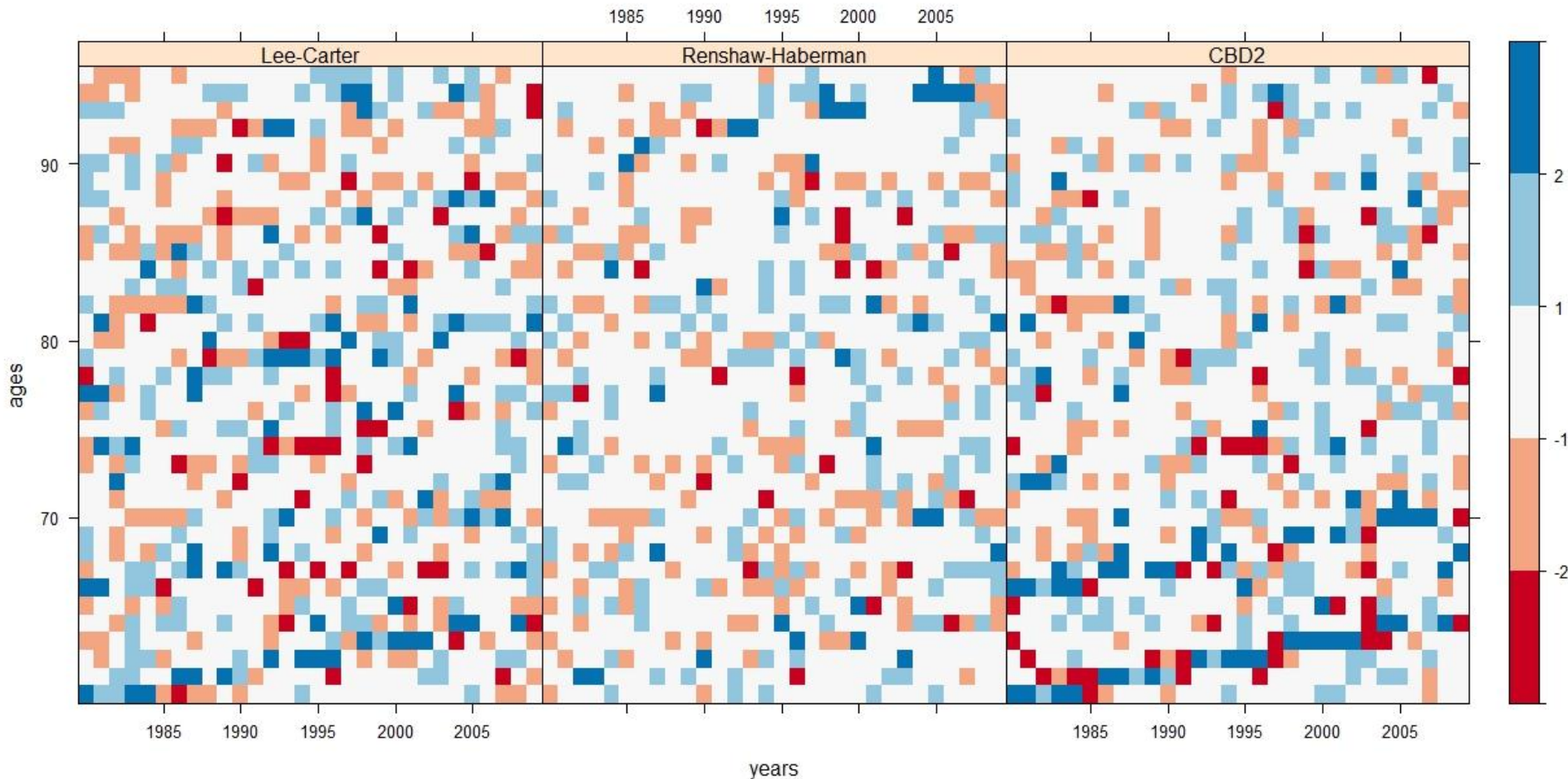
Femmine



# Confronto tra i modelli: residui standardizzati

Maschi

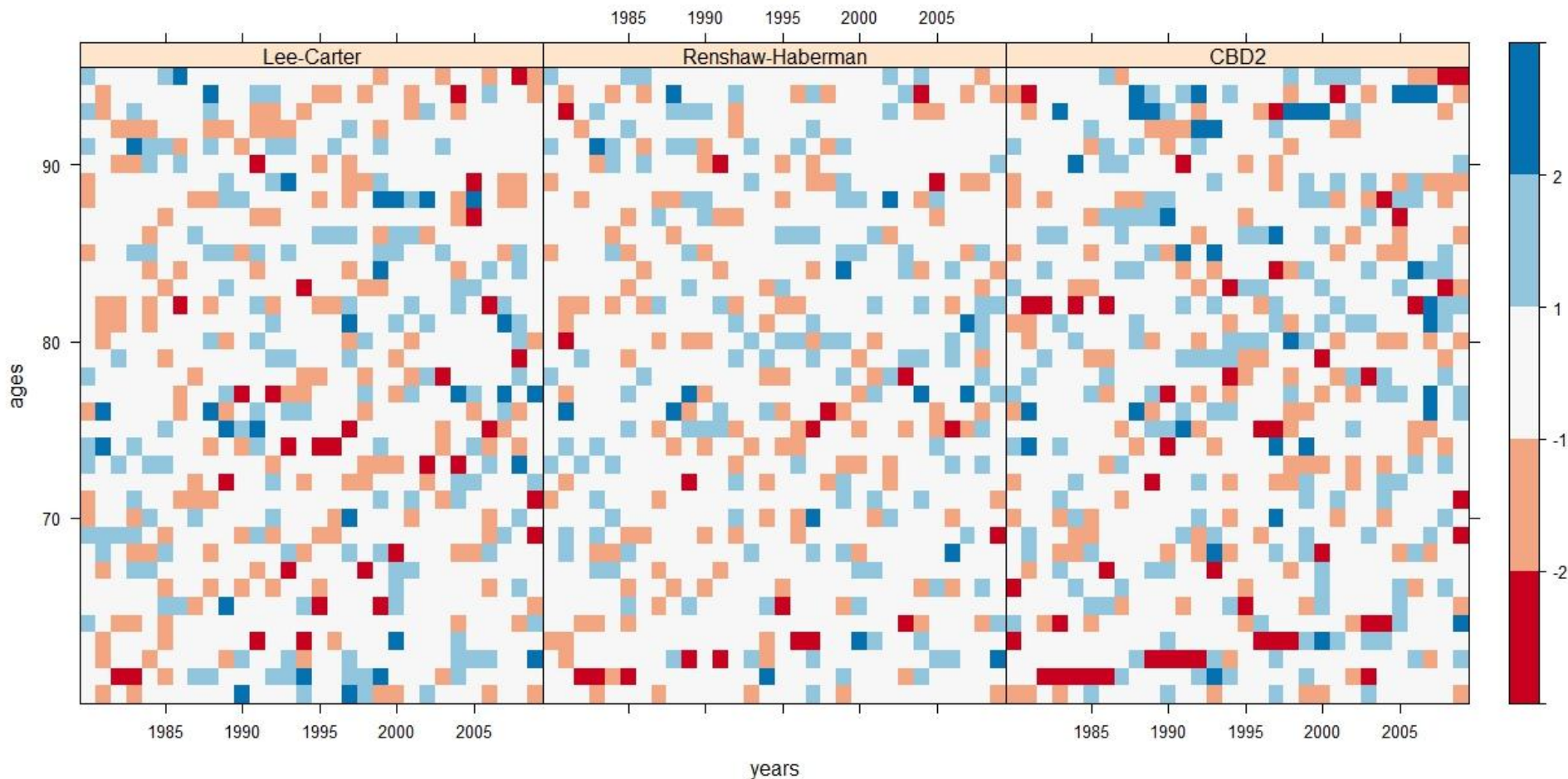
INPS autonomi



# Confronto tra i modelli: residui standardizzati

Femmine

INPS autonomi



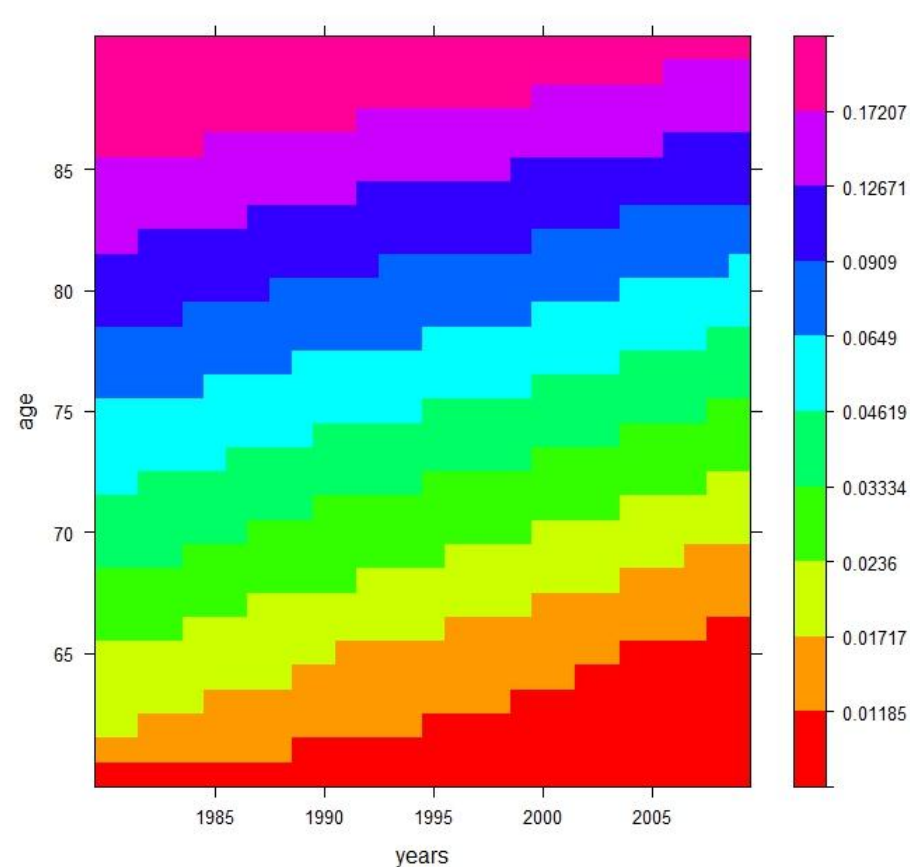
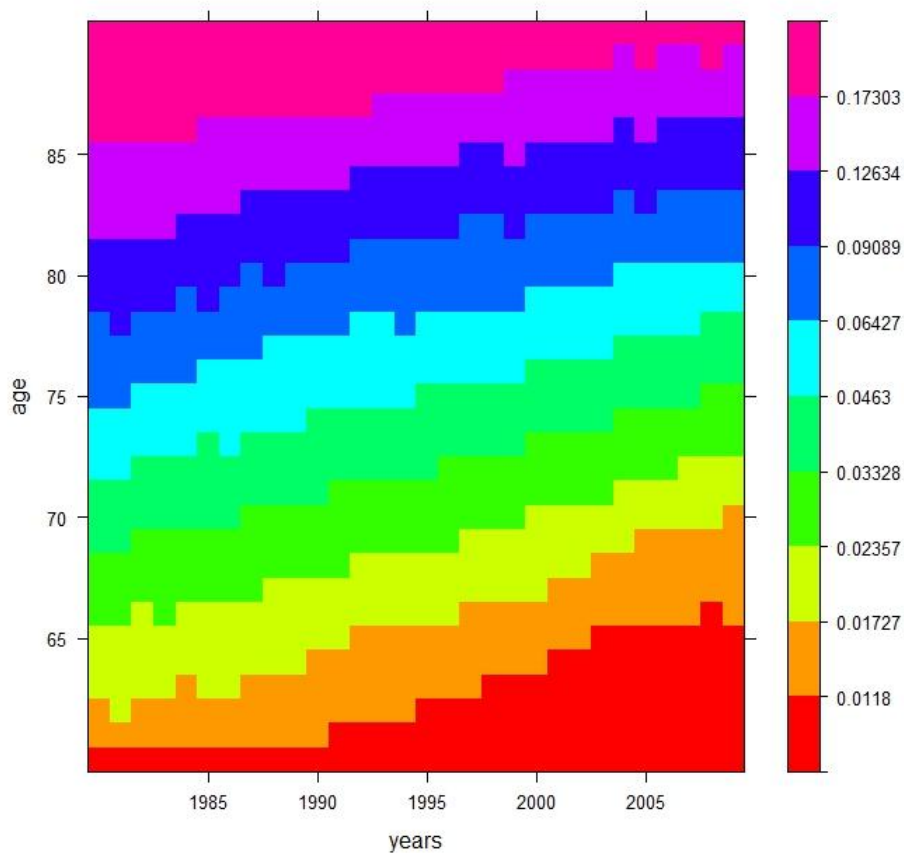
# INPS: lavoratori totali

# Tassi centrali di mortalità: **Inps tot.**

Maschi

$m_{x,t}$  – dati **grezzi**

$m_{x,t}$  – dati fittati con il modello **Lee-Carter**

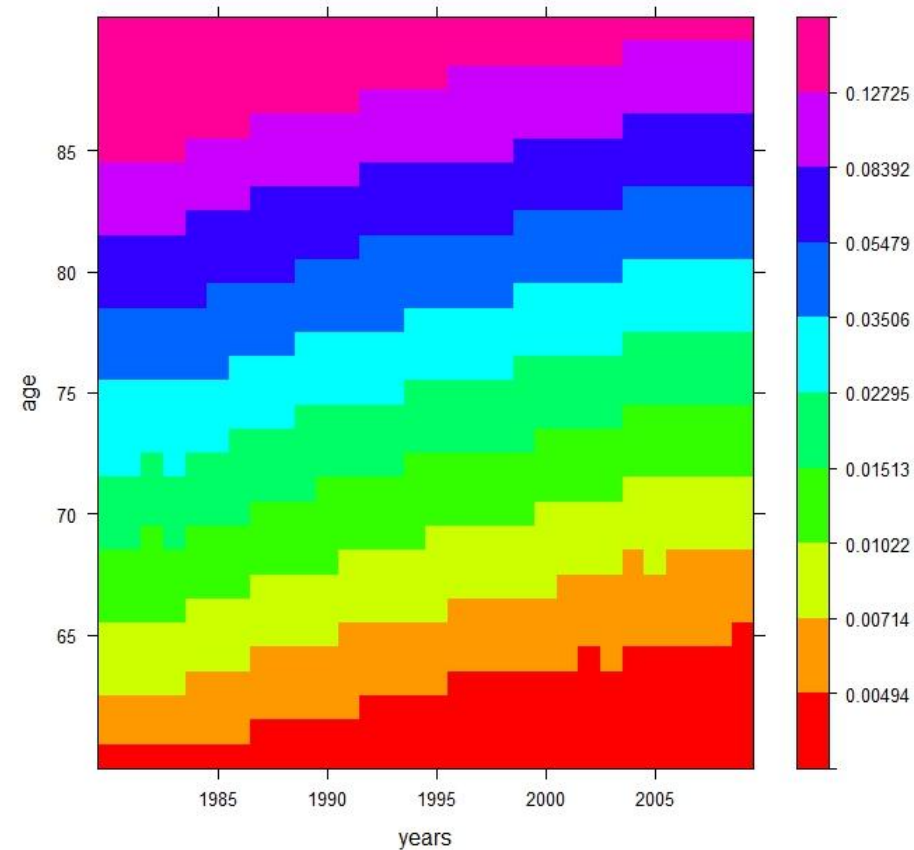
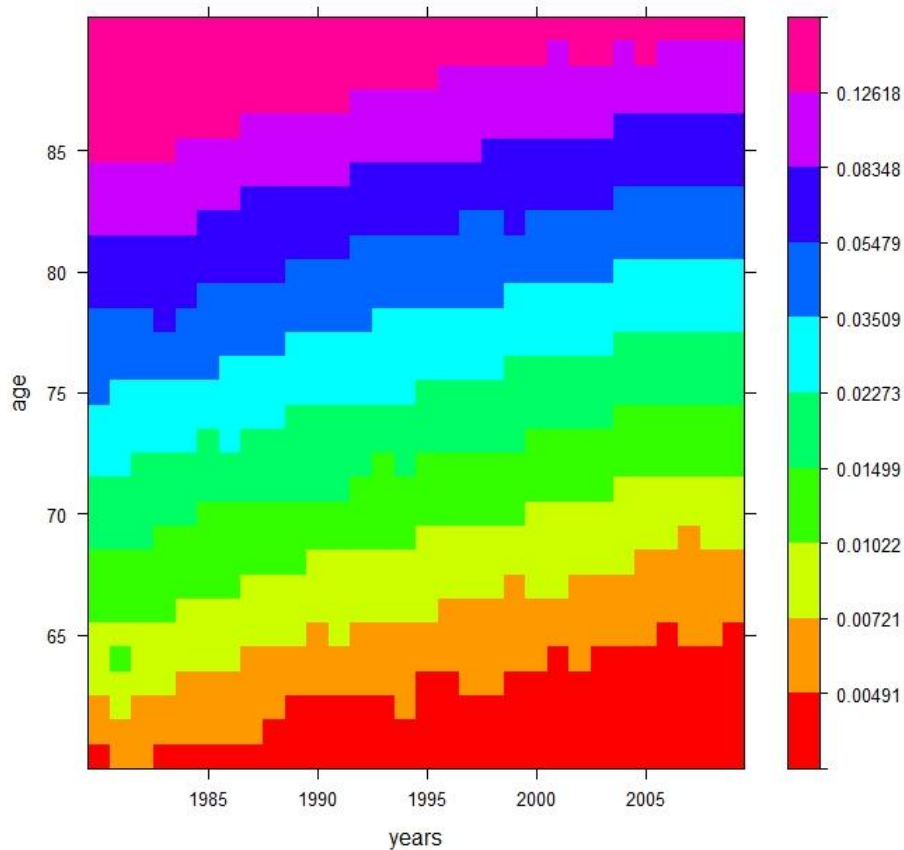


# Tassi centrali di mortalità: **Inps tot.**

Femmine

$m_{x,t}$  – dati **grezzi**

$m_{x,t}$  – dati fittati con il modello **Lee-Carter**





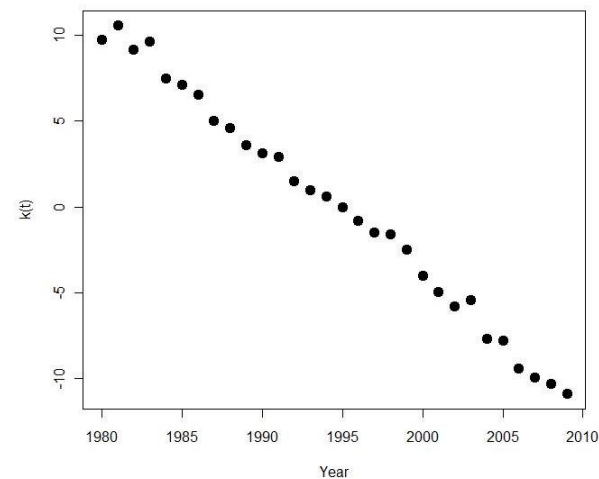
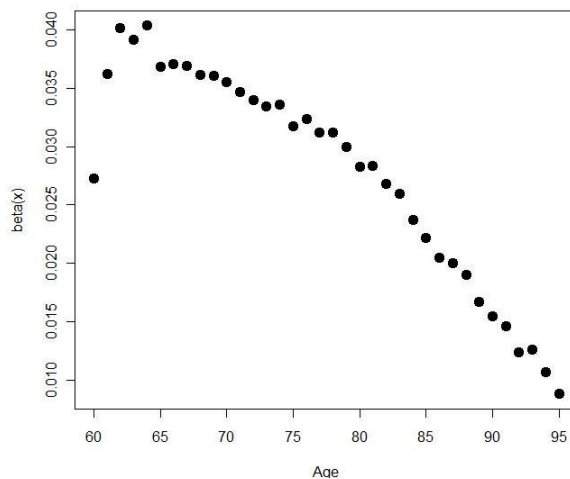
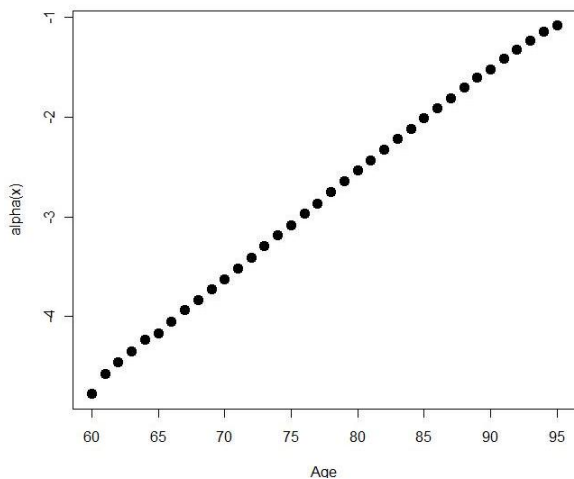
# Parametri del modello Lee-Carter: **Inps tot.**

Maschi

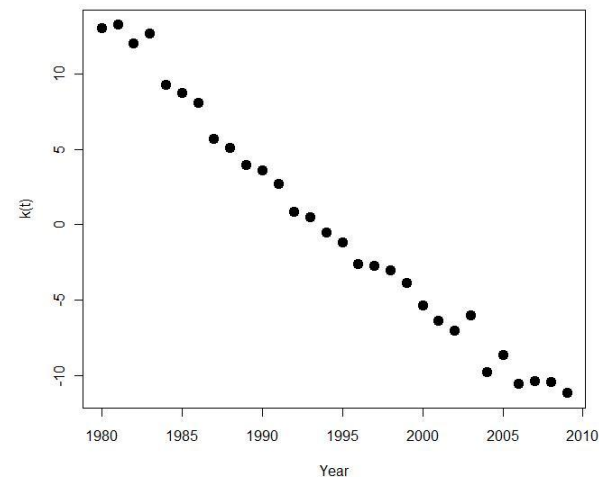
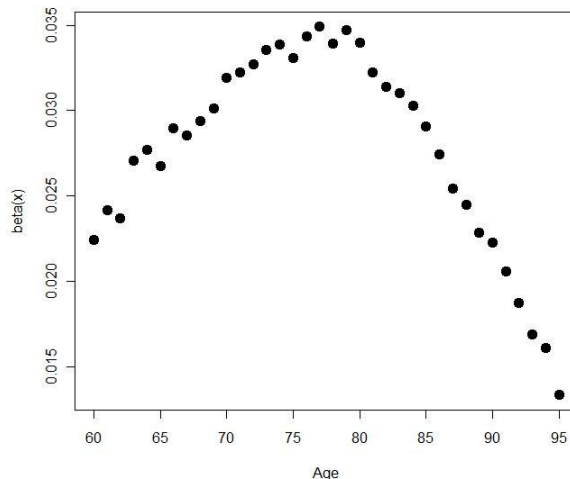
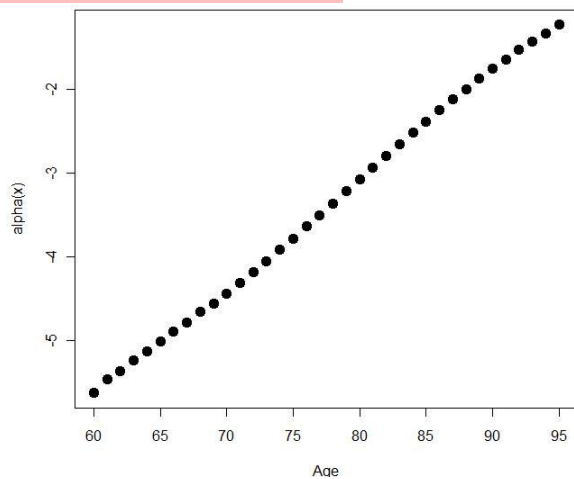
Beta 1

Beta 2

Kappa 2



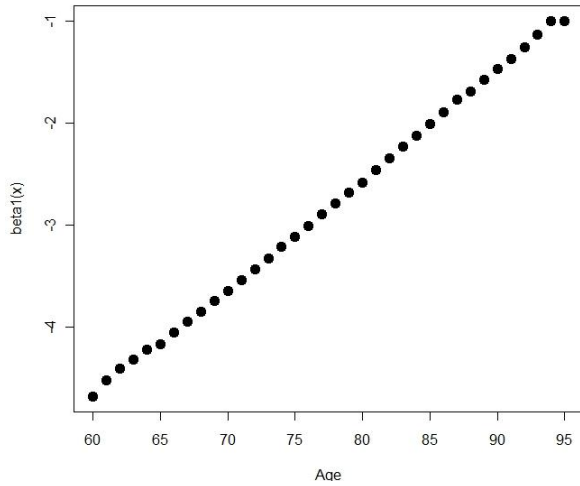
Femmine



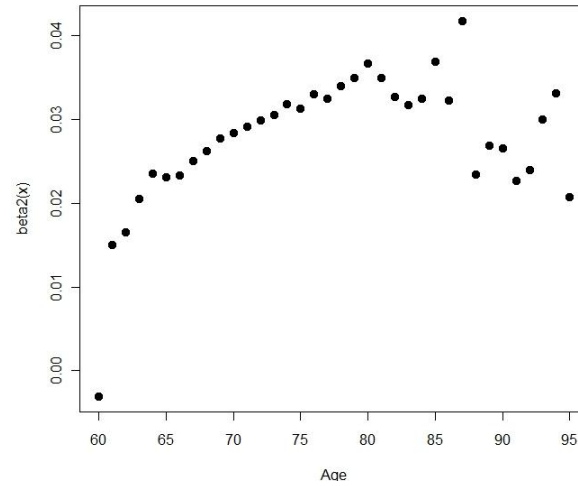
# Parametri del modello RH: **Inps tot.**

Maschi

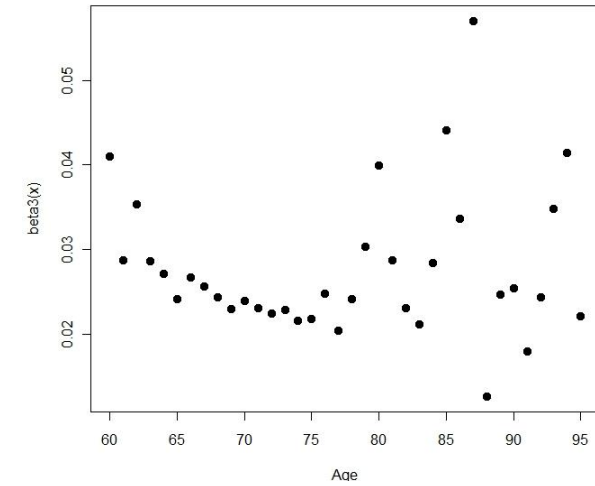
Beta1



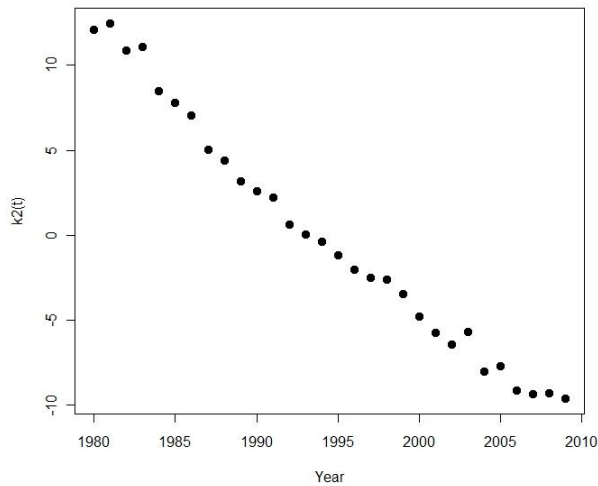
Beta2



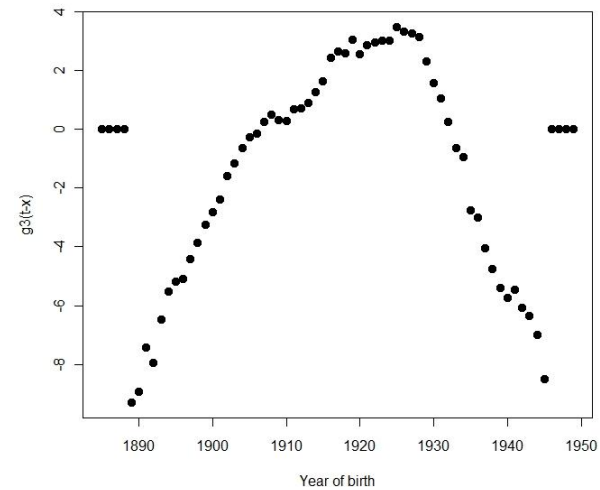
Beta3



Kappa2



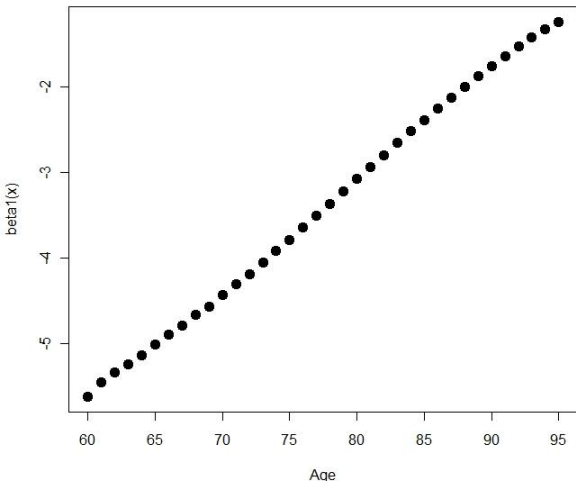
Gamma3



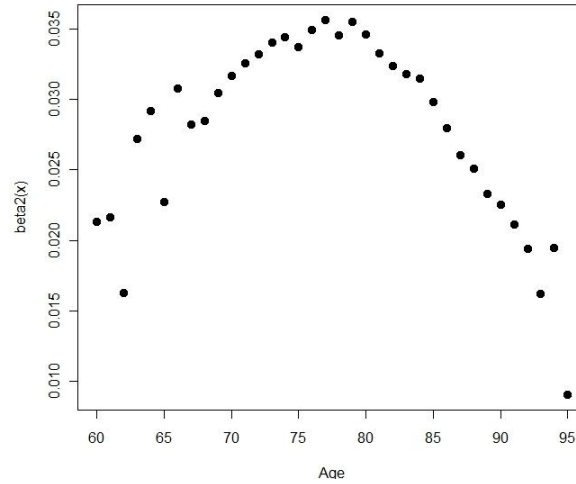
# Parametri del modello RH: **Inps tot.**

Femmine

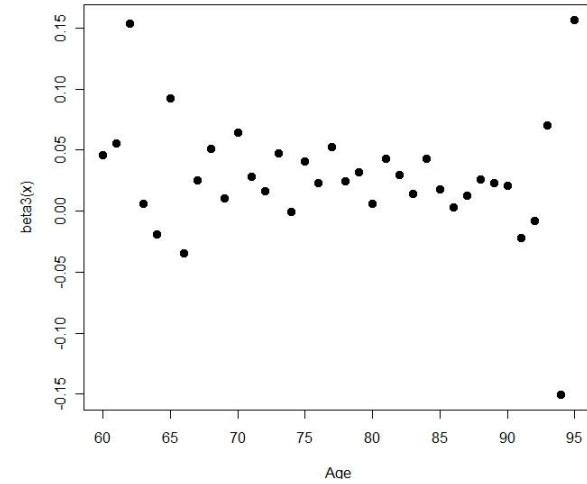
Beta1



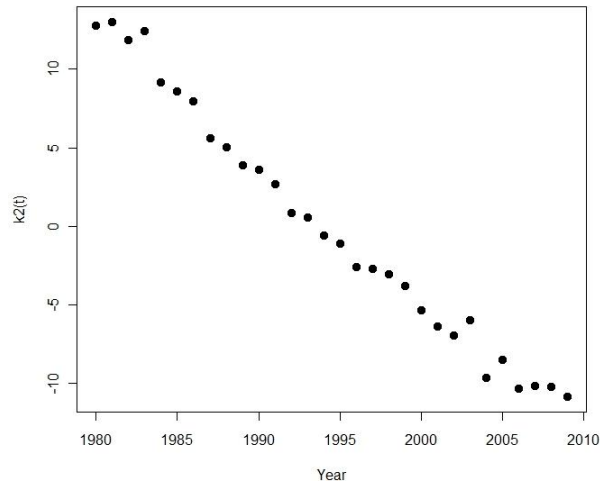
Beta2



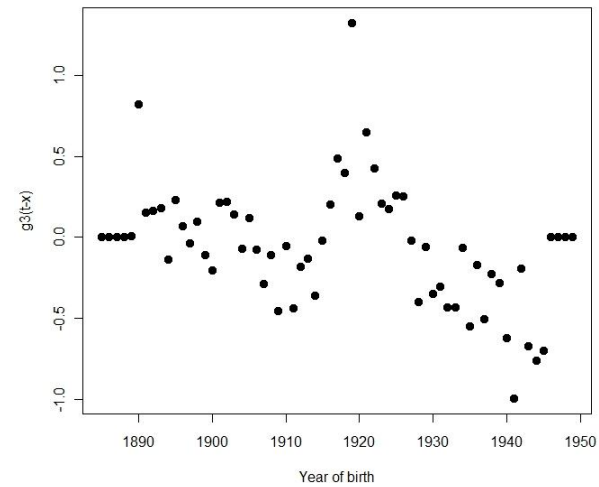
Beta3



Kappa2



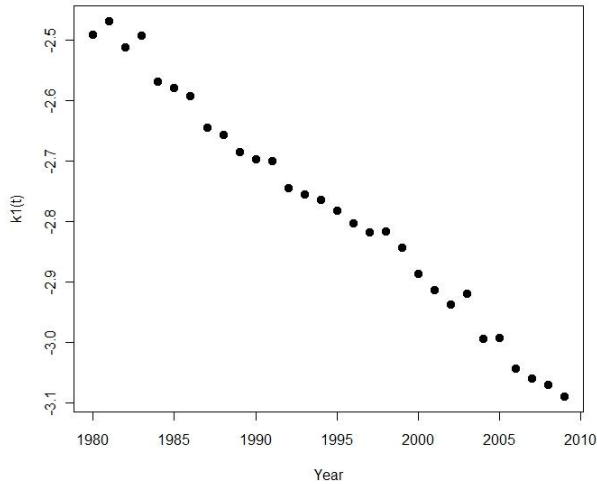
Gamma3



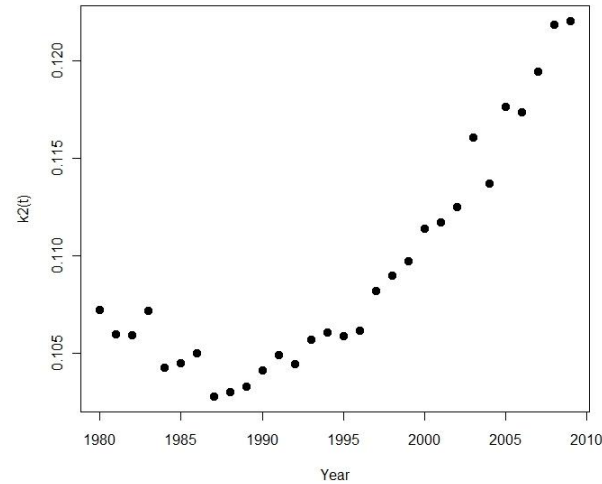
# Parametri del modello CBD-2: **Inps tot.**

Maschi

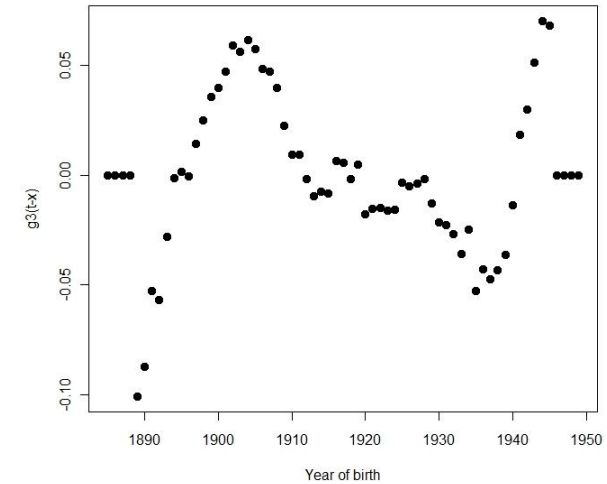
Kappa1



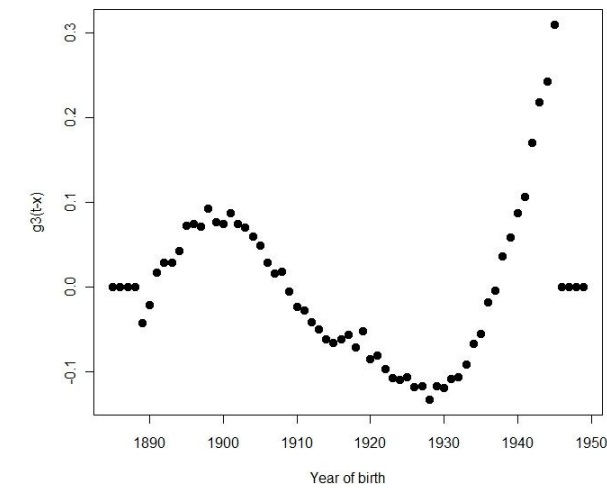
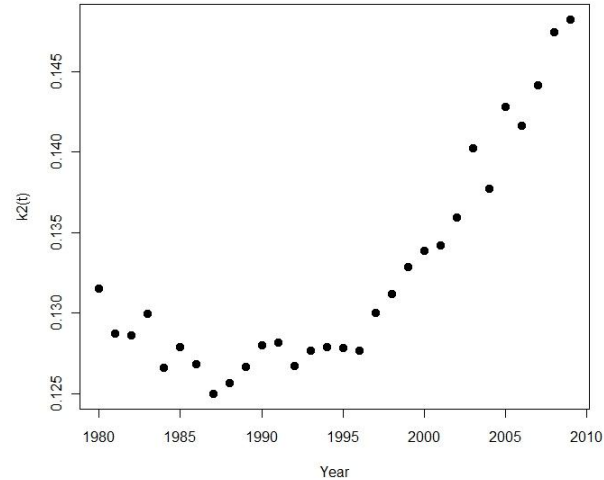
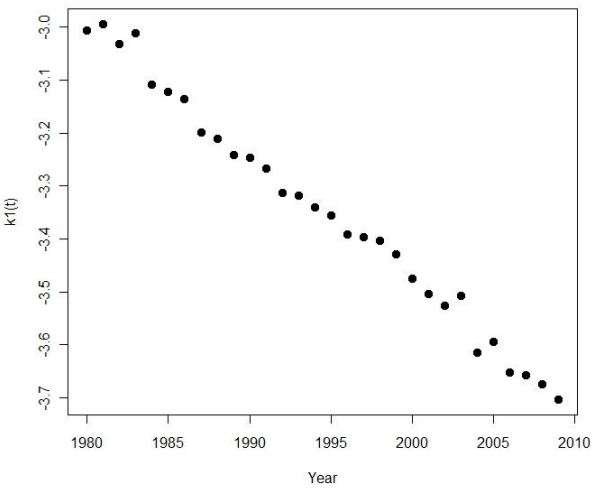
Kappa2



Gamma3



Femmine

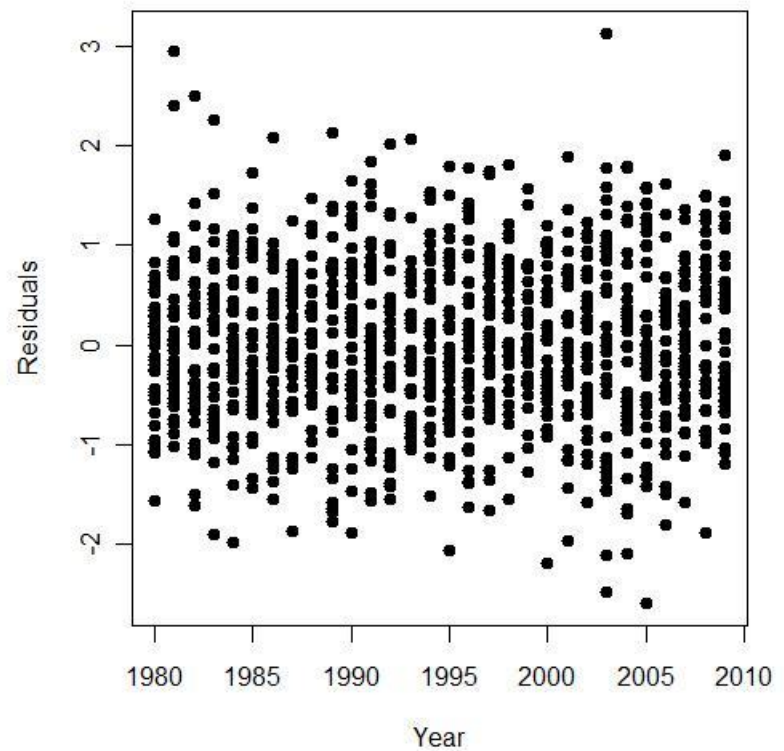
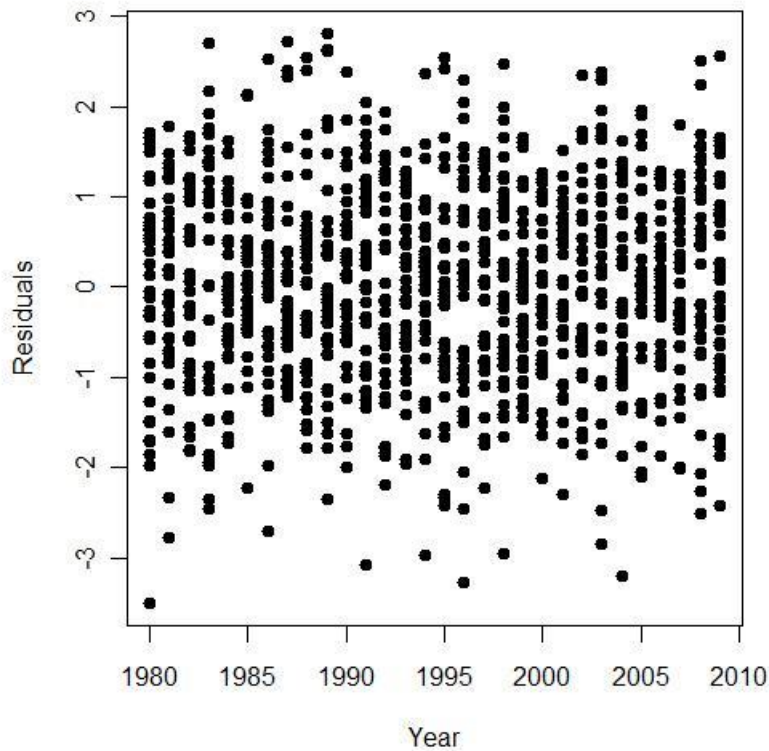


# Modello Lee-Carter: residui standardizzati

Maschi

INPS totale

Femmine

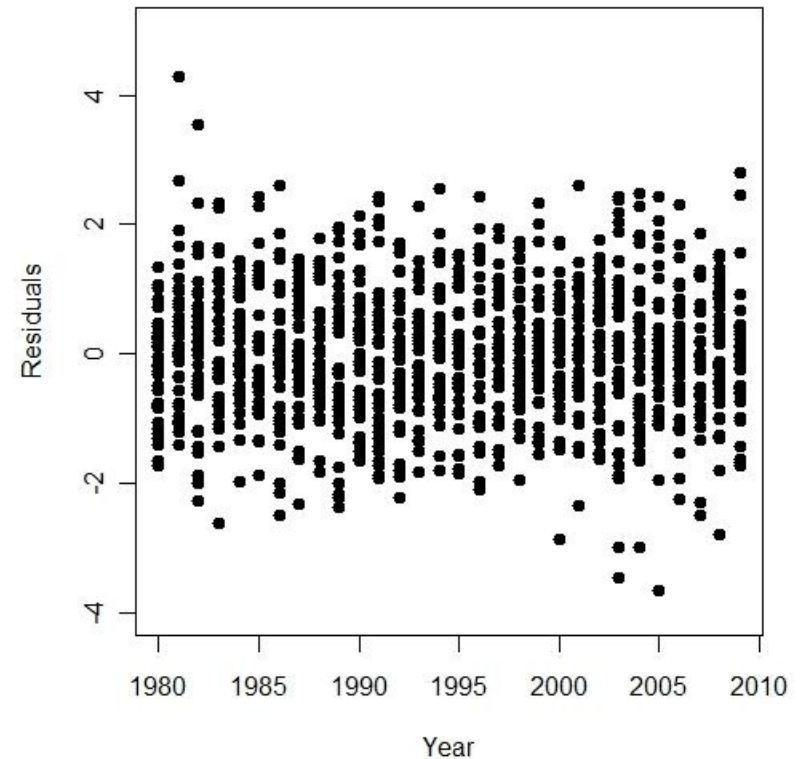
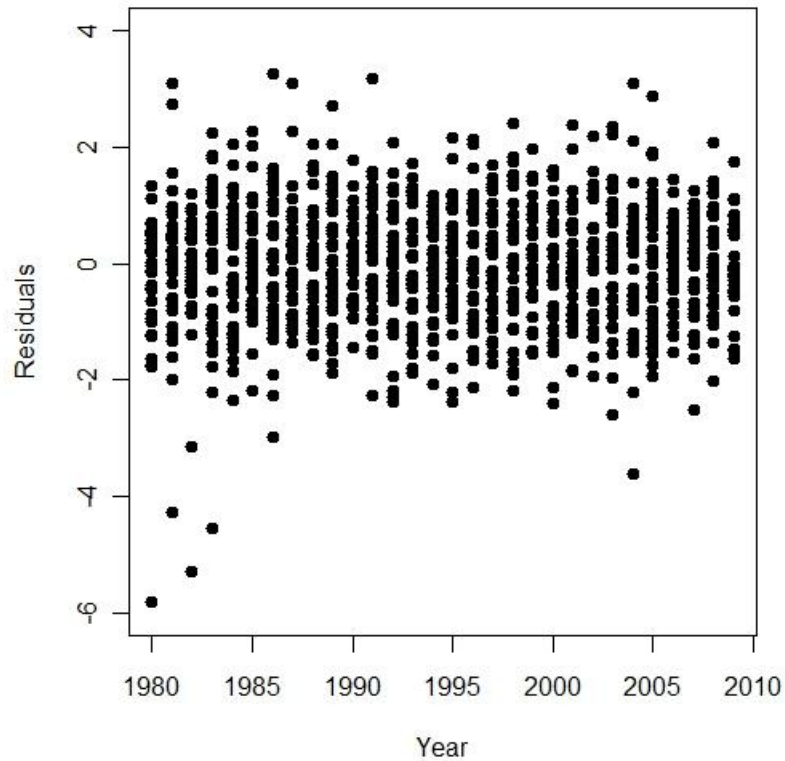


# Modello RH: residui standardizzati

Maschi

INPS totale

Femmine

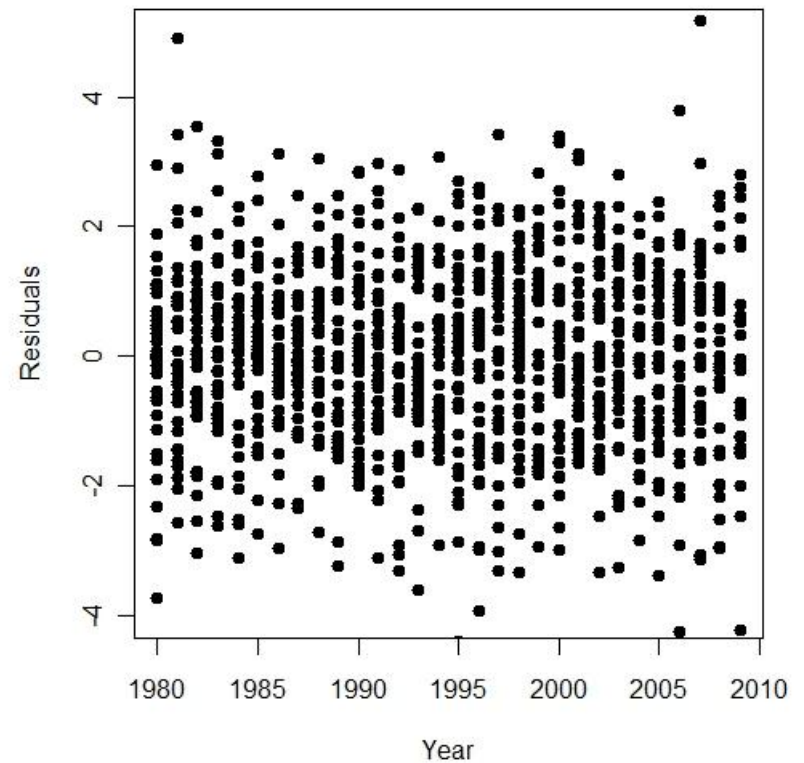
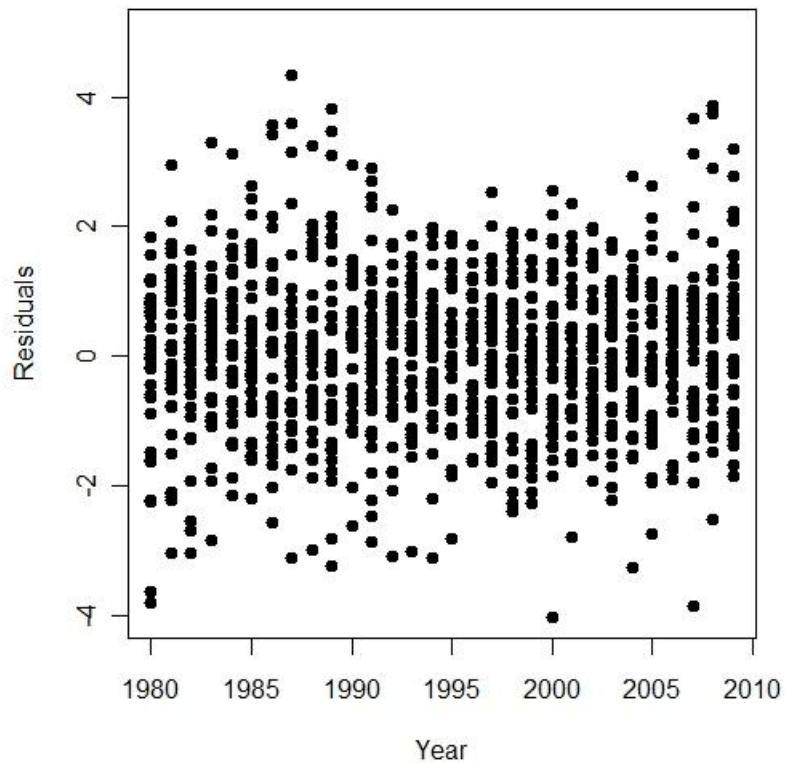


# Modello CBD-2: residui standardizzati

Maschi

INPS totale

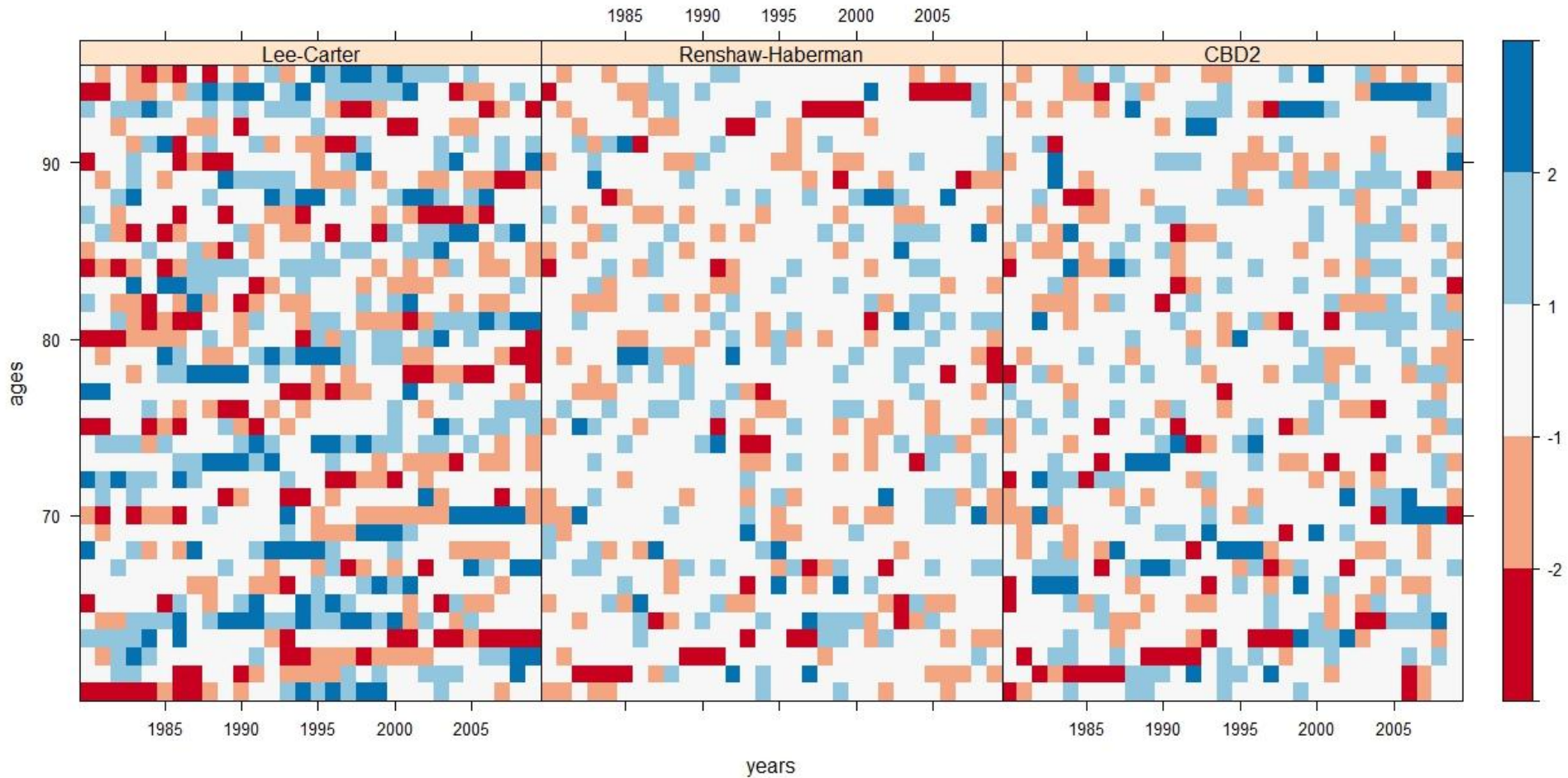
Femmine



# Confronto tra i modelli: residui standardizzati

Maschi

INPS totale

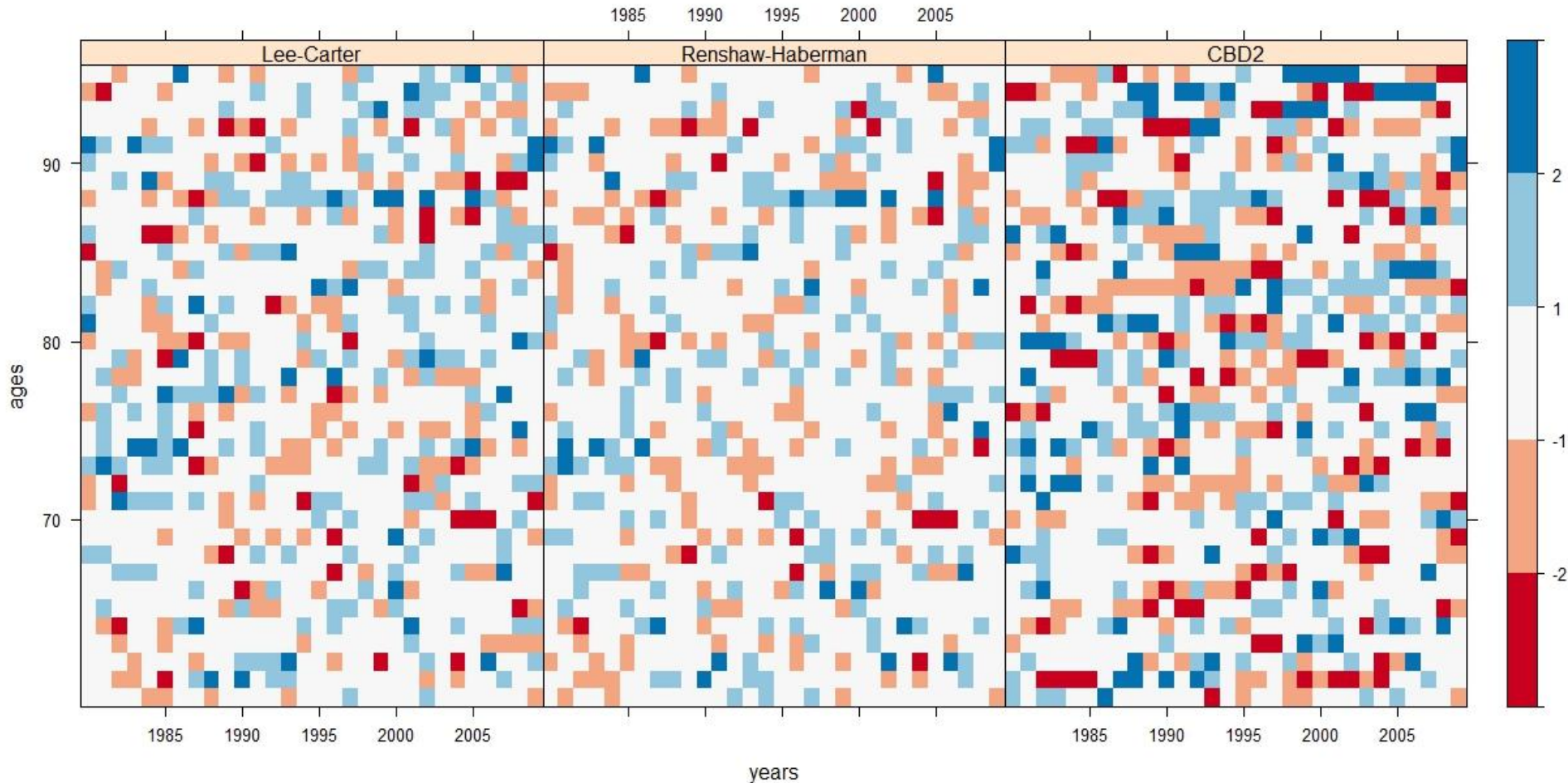




# Confronto tra i modelli: residui standardizzati

Femmine

INPS totale



**FINE**